2080

CHARGED PARTICLES IN EXTERNAL ELECTROMAGNETIC FIELDS

A GROUP THEORETICAL APPROACH

N. GIOVANNINI

CHARGED PARTICLES IN EXTERNAL ELECTROMAGNETIC FIELDS

A GROUP THEORETICAL APPROACH

PROMOTOR:
PROF. Dr. A. G. M. JANNER

CHARGED PARTICLES IN EXTERNAL ELECTROMAGNETIC FIELDS

A GROUP THEORETICAL APPROACH

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN
DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE
KATHOLIEKE UNIVERSITEIT TE NIJMEGEN, OP GEZAG
VAN DE RECTOR MAGNIFICUS PROF. DR. A. J. H. VENDRIK
VOLGENS BESLUIT VAN HET COLLEGE VAN DECANEN
IN HET OPENBAAR TE VERDEDIGEN OP
DONDERDAG 7 OCTOBER 1976
DES NAMIDDAGS TE 2 UUR PRECIES

door

NICOLAO PAOLO DAVIDE GIOVANNINI

geboren te Lausanne

1976
Druk: Krips Repro, Meppel

Dankbetuigingen

Ik wil hier graag mijn erkentelijkheid betuigen aan allen die aan de totstandkoming van dit proefschrift hun medewerking hebben verleend met suggesties en nuttige opmerkingen, maar waarvan de meesten op grond van artikel 23 van de promotieregeling hier niet expliciet genoemd mogen worden.

Mijn dank gaat ook naar Mevr. Riet van Kempen, die met grote zorgvuldigheid en bekwaamheid het vele en moeilijke typewerk heeft verzorgd en naar het personeel van de drukkerij Krips Repro, dat dit proefschrift met grote zorg heeft behandeld.

Het is voor mij tenslotte een groot genoegen alle personen binnen en buiten de Katholieke Universiteit van Nijmegen hartelijk te bedanken voor de sympathieke gastvrijheid die ze tijdens mijn verblijf in Nijmegen hebben getoond.

CONTENTS

GENERAL INTRODUCTION

This thesis contains a general group theoretical analysis of the problem of a charged massive particle moving in an external classical electromagnetic (e.m.) field, starting from the general symmetry properties of e.m. fields and of e.m. field equations. In this introduction we first want to sketch and comment on some of the background of the problem and on the guiding lines of our approach.

The main guiding line of the present work can in fact be seen in the word symmetry itself, which originates from the greek σύν (with) and μέτρον (measure) and whose basic meaning is harmony, equilibrium and regularity of proportions. That symmetry has played and plays in modern physics such an important role is then easily explained: symmetry lies in fact at the root of the concept of science, as science is essentially concerned with the search for the regularities of the laws of nature as hidden in measurable (and often apparently uncorrelated) events. As a matter of fact, in many problems and in a large domain that ranges from elementary particle classifications up to quantum chemistry, group theory, which is the natural mathematical structure generated by a set of symmetries, has led to and still leads to important and useful developments. Not only the symmetry of a physical law can thereby serve to clarify and analyse the properties of a physical system, but also information on the form of the physical law itself can be obtained from general invariance principles. In fact, as emphasized especially by Wigner, both in his immediate scientific work and in his more epistemological reflections (see e.g. [1,2]), invariance principles may provide "superlaws" that restrict the possible form of the dynamical laws obeyed by physical systems and serve as a guide for finding them when they are unknown. Conversely, when these laws are well accepted, invariance principles may throw new light on them, reveal new relations or eliminate apparent arbitrariness [1-3]. In this thesis we shall in fact be concerned with problems at both of these levels: we shall derive and discuss properties first of <u>invariance</u> operator groups that leave an equation of motion invariant (and that are based on space-time symmetries of e.m. fields), and second of <u>covariance</u> operator groups that leave the form of the equation invariant (and that are based on the space-time symmetries of the field equations). In both cases these operator groups will consist of kinematical (space-time) transformations combined (in general in a non-trivial way) with dynamical (gauge) transformations; their construction will be based on the well known fact that charged particles moving in an e.m.field are usually described by equations of motion that depend on the given field only through an electromagnetic potential.

The choice of the problem treated here rests in particular on the fact that electromagnetic interactions are quite well understood and at the same time very rich and informative, so that this problem emerges in a very natural way as part of a more general attempt to study and describe interacting systems from the point of view of their symmetry properties. We also hope that, in this way, standard applications of group theoretical methods in physics can then give rise to useful generalizations: the fact that for instance e.m. fields may have interesting and non-trivial symmetries in space and time can be used to investigate generalizations to time dependent problems of the well known and successful methods based on space symmetries and developed for time independent non-relativistic quantum mechanics. On the other hand, the successful description of elementary free particles as continuous PUAIR (projective unitary/antiunitary irreducible representations) of space-time kinematical covariance groups (in the relativistic [4] as well as in the non-relativistic [5] cases and with the Poincaré and the Galilei groups,respectively) can also usefully be extended to the case where an e.m. field is present.

When a charged particle interacts with an e.m. field it is customary to use the so-called external field approximation, principally because of its simplicity: the e.m. field is supposed to be uninfluenced by the self field of the particle. In this way the field equations (the Maxwell equations) decouple from the matter-field equations (equations of motion) and attention can be concentrated on the latter. Even if it is in principle well understood how the interaction can then be described (especially since the basic papers of Schwinger and of Salam and Matthews[6]), basic problems remain open, as we shall see below. On the other hand, the surprising general power of single-particle approximations justifies from another point of view a detailed analysis of the single-particle problem.

In the last few years, charged particles interacting with external e.m. fields have again been the subject of increasing attention,this for various reasons. The availability of very large magnetic fields in laboratories,for example,as well as the large fields (up to 10^{12} gauss) that are thought to occur in astrophysics have stimulated much research in this direction. At a more fundamental level, the belief (as emphasized for example by Wightman [7]) that the external field problem seems to be a first,necessary and probably very useful step towards a full quantized theory, has also motivated many new investigations. Another fact that should also be mentioned here is the recent discovery of unexpected difficulties in problems of this kind: first of the occurence in a very simple quantum mechanical system (namely the problem of a Bloch electron in a constant uniform magnetic field) of groups that are mathematically very hard to handle and physically also difficult to interpret: the quite pathological non-Type I groups [8]. Second is the rather unexpected fact discovered by Velo and Zwanziger [9,10] that beside the already known difficulties arising from conflicts between the dynamical equations and the constraint conditions, the usual minimal coupling with the external field may lead (for spins larger or equal to 1), to a-causal propagators. This means that, when they exist, the solutions of these equations of motion then correspond to a faster-than-light propagation of the particle, i.e. in terms of field operators, the fields satisfying these equations are not local [11]. This phenomenon has been shown to be essentially due to the (in fact non-natural) presence of supplementary components, i.e. of constraints. The very existence for example of the metastable Ω^- particle (whose spin is believed to be 3/2) shows that this problem is a more than academic one.

Having outlined some of the global physical interest and relevance of the problem, let us now explain how and why a group theoretical approach may be helpful in this context and let us give a general survey of the questions tackled in this thesis. The equation of motion for a charged particle moving in an external e.m. field is generally obtained from the corresponding equation of motion of the free particle by replacing the 4-momentum operator p by $p - \frac{e}{c} A(x)$, where e is the charge of the particle, c is the speed of light and A is some 4-potential corresponding to the external field. This recipe, called minimal coupling, is applied on analogy with the corresponding classical action principle [12] and, except for the above mentioned difficulties, has generally revealed itself as a very useful and successful one. There are however some apparently trivial but important consequences that follow from the change occured in the equation of motion. First, as is well known, there is not just one potential for a given field, but an infinite set of them since any gauge transformation

$$A(x) \longrightarrow A(x) + \partial\chi(x) \quad ,$$

for any real differentiable function on space and time $\chi(x)$, leaves the field and hence the physical system invariant. This is not only related to the fact that the potential is not an observable (as the field is) but it also implies that the transformation law of a potential under a space-time group element contains some arbitrariness. On the other hand it is not, in general, possible to choose a potential in this set in such a way that it has the same space-time symmetry group as the field, as in the simplest example of a constant uniform e.m. field. Consequently the space-time operators which correspond to these symmetries of the field (and thus of the physical system) do not in general commute with the operator of the equation of motion considered. It is however possible to use the arbitrariness mentioned above and to combine gauge and space-time transformations in such a way that for each symmetry element of the field there is a combined transformation which does leave the potential invariant; it is then possible to construct on this basis operator groups which do commute with the operators of the equations of motion. These groups are then called <u>invariance operator groups</u>. The principle of this method, which is in fact quite old, was used for example by Zak and by Brown [13] (on the basis also of previous investigations by other authors, e.g. by Johnson and Lippman, by Harper and by Fischbeck [14]) to construct invariance operator groups for a non-relativistic Bloch electron in a constant uniform magnetic field. On a more general basis, this method has then been analysed by Jansen and Boon [15], and by Janner and Janssen [16]. These last authors proved in particular that, for an arbitrary field, the resulting groups do not depend essentially on the choice of a corresponding potential as different potentials related to the same field give rise to isomorphic groups.

In the first part of this thesis (Chapter I to III) we derive for (almost) arbitrary e.m. fields, the explicit form of these operator groups and we analyse in more detail their structure and their properties, in the relativistic as well as in the non-relativistic frames. Let us mention here that a particular characteristic of our approach is then that we do not restrict ourselves to symmetry groups that are either connected Lie groups of discrete groups (as is quite usual in such problems). We show on the contrary, that these two kinds of symmetries are in general interrelated in a way that may be far from trivial. This point is also illustrated at the hand of examples. It is perhaps also of interest to note here that the invariance operator groups obtained in this way may of course always be seen as realizations of projective representations of the space-time symmetry groups of the fields considered. This identification is however in a way superficial and is less informative than the approach that we shall follow: the classes of factor systems that occur are for example not arbitrary on these symmetry groups but are uniquely determined by the fields themselves as we shall see. Furthermore, even when these factor systems are known, the corresponding operator groups still depend on the explicit gauge transformations which occur and which are not arbitrary, even if they generate the correct factor system.

One of the consequences of this first part is that the invariance operator groups so constructed are in general - because of the occurence of non-trivial factor systems - not subgroups of the space-time covariance operator groups for the corresponding equation of motion (the latter being projective representations of the Poincaré or of the Galilei group respectively). Since symmetry (which relates by definition <u>identical</u> physical systems) is necessarily a particular case of covariance (which relates by definition equivalent physical systems), the concept and explicit form of covariance can be usefully extended to the case where an external field is present. Just as there is an invariant transformation corresponding to any space-time symmetry of the external field there ought to be a covariant transformation related to any space-time symmetry of the field equations. This is of course also closely related to the group theoretical meaning to be attached to the equation of motion of a particle minimally coupled to an external field. The relationship between the equations of motion of free particles and covariance principles has been well known since the pioneer work of Wigner [4] in the relativistic case and of Lévy-Leblond [5] in the non-relativistic case. Because of the presence now of the external field this relationship is no longer necessarily the same. These considerations are at the basis of the results presented in Chapter IV, where such a covariance group is made explicit (for the relativistic case), independently of any equation of motion, but essentially based on the Poincaré invariance of the Maxwell equations. In Chapter V the representations of this group are then analyzed and the relationship with the equations of motion and with the group theoretical concept (defined in a way analogous to the free case) of elementary particles in interaction with an external field is discussed. In the special case where the external field vanishes, all these relations are shown to reduce to the usual ones. It is, in particular, indicated how our approach then leads to a possible explanation of the higher spin inconsistencies mentioned above. The Klein-Gordon and the Dirac equations, however, minimally coupled with an (almost) arbitrary e.m. field are shown to correspond effectively to representations of this new covariance group, i.e. to transform covariantly. In this way a new relationship, analogous to that in the free case, is found between these equations and a covariance statement. Furthermore, the invariance operator groups discussed before, which are generated by definition by those covariant transformations leaving the external field invariant, appear then in a natural way as subgroups of the general covariance operator group representations characterizing the corresponding equation of motion - i.e. as ordinary (and no longer projective) representations of subgroups of the covariance operator group of that equation.

Without it being the original intention, the subject covered in the present study has finally turned out to be quite a general one in so far as it deals in principle with any external field problem. One could then of course argue that, as a consequence, few concrete physical results will emerge. Things are, however, not necessarily simple as that, and what are apparently formal developments may also lead to helpful (and sometime even essential) new insights. On the other hand, specific physical problems do emerge in a natural way from our analysis as examples of the more general properties discussed, and are sometimes seen in a promising new light.

Let us finally mention here that this thesis is essentially constructed on the basis of four papers planned for publication (respectively the Chapters I, IV, V and the Appendix VA). As a consequence there is some unavoidable repetition for which we apologize; also many detailed calculations are omitted or made as short as possible in the hope of achieving a better global clarity.

Notes and References

[1] E.P. Wigner, Symmetries and Reflections, Indiana Univ. Press Bloomington & London, 1967

[2] E.P. Wigner, Physics Today 17, 34 (1964)

[3] J.M. Lévy-Leblond, Rev. Nuovo Cim. 4, 99 (1974)

[4] E.P. Wigner, Ann. Math. 40, 149 (1939)

[5] J.M. Lévy-Leblond, Thesis Orsay, 1965, and in "Group Theory and its Applications", Vol. 2 (ed. E. Loebl) N.Y. Acad. Press 1971 p. 221 (see also ref. in there)

[6] J. Schwinger, Phys. Rev. 93, 615 (1954)
A. Salam and P.T. Matthews, Phys. Rev. 90, 690 (1953)
(see also Thirring "Principles of Quantum Electrodynamics" N.Y. Acad. Press, 1958)

[7] A.S. Wightman, Proceedings Fifth Coral Gables Conference (eds. A.Perlmutter, C.A. Hurst, Freeman & Co, San Francisco 1968) (see also R. Seiler in [10])

[8] M. Boon, J. Math. Phys. 13, 1268 (1972)

[9] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969), 188, 188, 2218 (1969)

[10] "Troubles in the External Field Problem for Invariant Wave Equations", Tracts in Math. and Nat.Sc.Vol.4, (rev.A.S. Wightman), Gordon and Breach, 1971. (see also the aboundant literature on the subject since the original papers [9])

[11] The problem seems actually to be independent of the electromagnetic character of the interaction: analogous troubles occur in presence of gravitational fields (see e.g. G.W. Gibbons, J. Phys. A, 9, 145 (1976)). This illustrates by the way the point of view that a good understanding of electromagnetic interactions may generally have a wider significance than as applied to electromagnetic problems only.

[12] see for instance Landau and Lifshitz, Cours de Physique Théorique, Vol. 2, Eds. Mir, Moscou, 1966

[13] J. Zak, Phys. Rev. 134, A1602 and A1607 (1964), 136,A776 (1964)
E. Brown, Phys. Rev. 133, A1038 (1964)

[14] M.H. Johnson and B.A. Lippmann, Phys. Rev. 76, 828 (1949)
P.G. Harper, Proc. Phys. Soc. (London) A68, 879 (1955)
H. Fischbeck, Phys. Status solidi 3, 1082 (1963)

[15] L. Jansen and M. Boon, "Theory of finite groups. Applications in Physics", North Holland Amsterdam, 1967 (see esp. Chap.III, §6)

[16] A. Janner and T. Janssen, Physica 53, 1 (1971)

CHAPTER I

INVARIANCE OPERATOR GROUPS FOR A CHARGED PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

Introduction

The use of symmetry properties and thus of group theoretical methods in the study of an interacting physical system has often shown itself in the past to be very fruitful. Such an approach may in fact greatly facilitate obtaining information about solutions of the equations of motion, selection rules, conserved quantities and so on. In the frame of a research program concerned with the study of possible physical systems with non-trivial space-time symmetries, a very natural problem is that of a massive charged particle moving in an external e.m. field. Simple more specific cases of such systems have been quite extensively treated in the past, with first of all the case of zero field in the celebrated paper of Wigner [1]. Let us mention in particular space periodic fields (solid state), constant uniform (c.u.) fields [2] and also time dependent problems such as in the case of the field of an e.m. plane wave [3,4]. It is perhaps worthwile noting here that whereas for these two last examples exact solutions of the equations of motion have been in principle well known for a long time, the group theoretical analysis has been able to give a much better understanding: think, for instance, of the electric charge superselection rule in the c.u. field case [2] or of the so-called "mass-shift" of a charged particle in the plane wave field [3,4]. We hope therefore that group theoretical methods will allow us to approach the physics of systems with other types of fields where exact solutions are far from being known.

In this chapter we consider the most general case of an external e.m. field and derive first some general properties of the space-time symmetry groups which may occur. Although the equations of motion we consider here (Schrödinger, Klein-Gordon, Dirac) are potential and not (explicitely) field dependent, as is well known, it is possible to construct on the basis of the symmetries of the field potential symmetry groups using the very natural concept of compensating gauge transformations, and from these groups to derive then explicitly invariance operator groups which commute with the operators of the equations of motion and which thus contain physical information. That these last steps are not trivial and that the latter groups may have a very different structure from that of the original space-time groups may be seen in the example of a Bloch electron in a c.u. magnetic field where the (abelian) translation symmetry group of the field can give rise to very pathological non-Type I (definition given later on) operator groups [5,6]. We show in particular in this chapter that this example is only a special case of a large class of fields with such a property. We shall also see, in other examples of this situation, that it is in fact independent of the specifically magnetic character of the field.

This chapter is organized as follows. In part one we define the class of fields we consider and analyze the structure of their relativistic symmetry groups in more detail. This part may appear tedious but is useful for what follows, and for practical purposes. In part two we define correspondingly symmetry groups of potentials. In parts 3 and 4 we then construct explicitly invariance operator groups for spins 0 and $\frac{1}{2}$ (Klein-Gordon and Dirac) equations and discuss some of their properties. The general theory is then used in part 5 for a closer analysis of the problem of a Bloch electron in an arbitrary constant uniform external field. Since we can do so with little extra effort, we cover the Schrödinger case as well, and extend briefly in part 6 the theory to the quantum mechanical Galilean case, too.

Let us also mention here that some of the present results have been previously presented elsewhere [7].

1. Symmetry groups of e.m. fields.

Before we analyze the properties of symmetry groups of e.m. fields, let us first give a more precise definition of what we admit as possible candidates for them. An admissible e.m. field is defined as a continuous differentiable map from the Minkowski space M(4) to the space of covariant antisymmetric (time pseudo-) real tensors $F_{\mu\nu}$ with the following three properties

(i) it satisfies Maxwell equations

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$

$$F_{\mu\nu}{}^{,\nu} = \frac{4\pi}{c} j_{\mu} \tag{1.1}$$

Because we make, from a physical point of view, no further restriction on the 4-current j_μ (not all currents are necessarily generated by moving charges, for example) the second set of equations can and will be considered as a definition for j_μ.

(ii) it can be expanded as a Fourier integral

$$F_{\mu\nu}(x) = \int d^4k\ \hat{F}_{\mu\nu}(k)\ \exp\ (ikx) \qquad (1.2)$$

i.e. we assume that the integral

$$\hat{F}_{\mu\nu}(k) = \left(\frac{1}{2\pi}\right)^4 \int d^4x\ F_{\mu\nu}(x)\ \exp\ (-ikx) \qquad (1.2)'$$

exists (in the sense of course of generalized functions).

(iii) it transforms under an element $g = (t,\Lambda)$ [1] of the Poincaré group P according to

$$(gF)_{\mu\nu}(x) \stackrel{\text{def}}{=} \varepsilon(g)\Lambda_\mu^\rho\Lambda_\nu^\sigma F_{\rho\sigma}(\Lambda^{-1}(x-t)) \qquad (1.3)$$

with $\varepsilon(g) = \text{sign}\ (\Lambda_o^o)$. This law defines immediately a group G_F, the largest subgroup of P consisting of elements g which satisfy the condition

$$(gF)_{\mu\nu}(x) = F_{\mu\nu}(x) \quad . \qquad (1.4)$$

---

1) We adopt the notation $g = (t,\Lambda) \in P$ with $(t,\Lambda)x = \Lambda x + t$, $x \in M(4)$, $t \in T$, the 4-translations subgroup of P, $\Lambda \in \mathcal{L}$, the homogeneous Lorentz group, $(\Lambda x)_\mu = \Lambda_\mu^\nu x_\nu$ for covariant, $(\Lambda x)^\mu = (\Lambda^{-1})^\mu_\nu x^\nu$ for contravariant vector components and $(t_1,\Lambda_1)(t_2,\Lambda_2) = (t_1+\Lambda_1 t_2,\Lambda_1\Lambda_2)$.

This condition depends on the origin and thus on the choice of the reference frame, but, as is well known, different choices give rise to isomorphic (conjugate) subgroups so that one can attach to any given external field a unique <u>abstract</u> symmetry group as the corresponding isomorphy class. The group defined by (1.4) is then to be seen as a representative of this class. This group has the following evident properties:

(1) Because of the continuity condition on the field, $G_F \subseteq P$ is a closed subgroup.

(2) $G_F \cap T \overset{\text{def}}{=} U_F \cong \mathbb{Z}^{r_o} \oplus \mathbb{R}^{m_o}$, $r_o + m_o \leq 4$, since U_F is closed in T and the only closed subgroups of T are of this form.

(3) $U_F \triangleleft G_F$, i.e. U_F normal in G_F, (since T is normal in P) and $G_F/U_F \overset{\text{def}}{=} \tilde{K}_F$ is isomorphic to a space-time point group K_F, i.e. to a subgroup K_F of the Lorentz group.

The group G_F appears thus as an extension of U_F by K_F and the following diagram of groups has exact rows and is commutative (morphism of group extensions) [1]

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U_F & \longrightarrow & G_F & \longrightarrow & K_F & \longrightarrow & 1, \quad m,\phi \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & P & \longrightarrow & \mathcal{L} & \longrightarrow & 1, \quad 0,\phi
\end{array}
\qquad . \qquad (1.5)
$$

The lower row splits, as is well known (semidirect product) whereas the first one is an (in general nontrivial) extension characterized by the factor set m

$$m: K_F \times K_F \longrightarrow U_F$$

and by the natural homomorphism ϕ of K_F into Aut (U_F). This means

---

1) througout this treatise $\rightarrowtail$ indicates a monomorphism, $\twoheadrightarrow$ an epimorphism.

that the product in G_F, with $g_i = (a_i, L_i) \in G_F$, $a_i \in U_F$, $L_i \in K_F$, $i = 1,2$, may be written as

$$(a_1, L_1)(a_2, L_2) = (a_1 + \phi(L_1)a_2 + m(L_1, L_2),\ L_1L_2) \quad .$$

Associativity of this product implies that m is a 2-cocycle (i.e. an element of $Z^2_\phi(K_F, U_F)$, i.e. satisfies, $\forall L_1, L_2, L_3 \in K_F$, the condition

$$m(L_1, L_2) + m(L_1L_2, L_3) = m(L_1, L_2L_3) + \phi(L_1)m(L_2, L_3)$$

or, in other words, m is a factor set. For more details and for the use of this cohomological language, we refer to [8].

As discrete symmetries will play an important role in the sequel it is also useful to consider here another decomposition of G_F as expressed in the following proposition (see also [9]):

Proposition 1.1. Let G be any closed subgroup of the Poincaré group, G^c its connected component of the identity, then G appears in the following commutative diagram of exact sequences

$$\begin{array}{ccccccccl} 1 & \longrightarrow & G^c & \longrightarrow & G & \longrightarrow & G^d & \longrightarrow 1, & n,\ \xi \\ & & \| & & \downarrow(\iota) & & \downarrow & & \\ 1 & \longrightarrow & G^c & \longrightarrow & N(G^c) & \longrightarrow & N(G^c)/G^c & \longrightarrow 1, & n',\xi' \end{array} \qquad (1.6)$$

where $N(G^c)$ is the normalizer of G^c in P, (ι) the canonical injective monomorphism and G^d is a discrete subgroup of $N(G^c)/G^c$. The factor system n and the map $\xi : G^d \longrightarrow \mathrm{Aut}\ (G^c)$ are the restrictions on $G^d \times G^d$ and G^d respectively of the factor system n' and map ξ' of the bottom extension.

Proof. (i) G^c is closed in G (thus in P) and normal in G, the latter follows from the definition of G^c and the fact that conjugation is a continuous automorphism, hence $g.g_1.g^{-1}$ is connected to 1 if and only if g_1 is (g being fixed).

(ii) G/G^c is discrete since the component of this group which is connected to the identity is 1 and is open.

(iii) The action of G^d on G^c being defined by conjugation in G after identification of G^d with a set of coset representatives of G/G^c, the proposition is then obvious.

Let us note here that the decomposition (1.6) may usefully be applied for the determination of the lattice of closed subgroups of the Poincaré group. Indeed connected subgroups may be classified by Lie algebra methods, using a well known theorem which states that if H is a Lie subgroup of a Lie group G, the Lie algebra of H is a subalgebra of the Lie algebra of G, and that, conversely, each subalgebra of the Lie algebra of G is the Lie algebra of exactly one connected subgroup H of G [9,10]. The normalizers may then in general be computed in a straightforward way and the problem reduces to the determination of the discrete subgroups of $N(G^c)/G^c$. Let us also note that this decomposition is of course not exclusive for the Poincaré group and might be useful in other cases as well.

We now want to consider more closely the discrete factor group G^d of (1.6), and mention first the following

Proposition 1.2. Let G as in Proposition 1.1. be a split extension in (1.6). Then G^d may be identified with a discrete subgroup of G and appears in the following exact sequence of groups

$$0 \longrightarrow U^d \dashrightarrow G^d \xrightarrow{\pi} K^d \longrightarrow 1, \qquad m^d, \phi \qquad (1.7)$$

where $U^d \cong \mathbb{Z}^n$, $n \leqslant 4$, and K^d is a discrete subgroup of the Lorentz group.

Proof. The proof of the exactness follows trivially from the hypothesis and the same arguments as in (1.5). That U^d and K^d are discrete follows from the fact that G^d is discrete and that the topology of U^d is the subset topology on one side and that the epimorphism π in (1.7) is continuous on the other side.

Let us now briefly show why this decomposition is useful and how difficulties may arise from this part of the top extension in (1.6). In fact very little is known on non-abelian discrete groups of infinite order and they are in general very difficult to handle. Let us mention in particular the following theorem that we shall need in the sequel.

Theorem 1.3 (Thoma [11]). A discrete group G is of Type I if and only if it has a normal abelian subgroup A of finite index.

More details about the pathological non-Type I groups (also termed wild groups) are given in the third part of this chapter.

Using this theorem one can in fact directly prove the following.

Proposition 1.4. Let G^d be any discrete subgroup of the Poincaré group (as in (1.7)), then G^d is of Type I if and only if there exists a normal abelian subgroup K_1 of K^d such that $\phi(K_1)$ is the identity on U^1, m^d is symmetric on K_1 and $|K^d/K_1| < \infty$.

Let us also mention here that if G^d is not of Type I then it is possible to show that G is also not of Type I whereas G^c in (1.6) being a connected real algebraic group, is necessarily always of Type I [12].

It is clear from the above proposition that G^d will only be of non-Type I if its point group K^d is of infinite order. It is therefore useful mentioning here without going into detail that this case seems to be of very questionable physical significance in our problem as there is no known example of e.m. field with this property (and presumably there exists none). This case will therefore just be mentioned for completeness in the sequel but it will not be considered too much in detail.

Because however of the important role discrete translational symmetries are going to play in the sequel it is useful to define here a <u>periodic e.m. field</u> as a covariant time-pseudo tensor field over $M(4)$ satisfying, in addition to the properties (i), (ii), and (iii) listed previously the following:

(iv) The spectrum S of F (which is defined as the set of points k in the reciprocal space-time $M^*(4)$ for which $\hat{F}_{\mu\nu}(k) \neq 0$ in (1.2)) is discrete and generates a lattice Λ^* (of dimension n_o, $o \leqslant n_o \leqslant 4$). In other words the field F can be written as a Fourier series over Λ^*:

$$F_{\mu\nu}(x) = \sum_{K\in\Lambda^*} \hat{F}_{\mu\nu}(K) \exp(iKx) \tag{1.8}$$

where clearly the connection to (1.2) is given by

$$\hat{F}_{\mu\nu}(k) = \sum_{K\in\Lambda^*} \hat{F}_{\mu\nu}(K)\, \delta(k-K) \quad . \tag{1.9}$$

In order to fix ideas let us give some examples of such periodic e.m. fields: constant uniform (c.u.) fields ($n_o = o$), monochromatic plane wave fields ($n_o = 1$, $K^2 = 0$), crystal fields ($n_o = 3$, $K^2 > 0$ $\forall K$ [1]), eigenmodes of waveguides or cavities and so on.

After this quite mathematical preliminary survey of properties of fields and their symmetry groups we pursue our way in the direction of physics by defining, for each given e.m. field, a symmetry group for a corresponding potential.

---

1) We use the metric $g_{\mu\nu}$ with signature $(-,+,+,+)$

2. Potentials symmetry groups.

As is well known Poincaré transformations which leave an e.m. field invariant are not, in general, symmetries for a corresponding potential and, as a consequence, the corresponding operators do not in general commute with the operators of the equations of motion we want to consider and which are, as is well known, potential and not (explicitely) field dependent (here the relativistic Klein - Gordon or Dirac, but also in the Galilean case with Schrödinger or classical equations with minimal coupling). As potentials are however not physical observables as fields are it is to be expected that the space-time symmetries of the field and thus of the physical system will reflect themselves in some way in symmetries of the potential and respectively of the equation of motion. On the other side there is a certain arbitrariness in the choice of the gauge and of the transformation law of a potential and one may take advantage of it. This occurs with the introduction of the quite natural concept of compensating gauge transformation. Let us resume briefly here, in a way convenient for our purposes, this idea as exposed by Janner and Janssen [13] :

Let $A_\mu(x)$ be some L-potential corresponding to a given e.m. field $F_{\mu\nu}(x)$, i.e. which satisfies

$$A_{\nu,\mu}(x) - A_{\mu,\nu}(x) = F_{\mu\nu}(x) \tag{2.1}$$

and which we shall assume to transform under the Poincaré group as a covariant (time-pseudo) vector field A, i.e., for $g = (t,\Lambda) \in P$

$$(gA)_\mu(x) = \varepsilon(g)\, \Lambda_\mu^\nu A_\nu(\Lambda^{-1}(x - t)) \tag{2.2}$$

with $\varepsilon(g) = \text{sign}\,\Lambda_o^o$ as in (1.3). Let now $g \in G_F$ be a symmetry of the field. Because it leaves the field invariant, it generates from A a new potential (gA) which can differ from A only by a gauge transformation (both A and gA being related to the same field) which may depend on g and on x:

$$(gA)(x) = A(x) + \partial \chi_g(x) \tag{2.3}$$

This gauge function is determined by (2.3) uniquely up to a constant. Combining now such a gauge transformation with each space-time element $g \in G_F$ we obtain pairs (χ_g, g) whose action on the potential we define by

$$(\chi_g, g)\, A(x) \overset{\text{def}}{=} (gA)(x) - \partial \chi_g(x) \tag{2.4}$$

In this way one has constructed a set of transformations $\{(\chi_g, g), \forall g \in G_F\}$ which do obviously leave the potential invariant. However, this set does not, in general, form a group because it is not necessarily possible to choose the gauge functions in (2.3) in such a way that the gauge function associated with the product of two Poincaré transformations is the gauge function resulting from the product of the corresponding two pairs. It is however possible to imbed this set in a group $\overline{Q}$ which is defined as follows:

$$\overline{Q} = \{ (\phi, \chi_g, g), \ \phi \in \mathcal{R}, \ g \in G_F \} \tag{2.5}$$

with product rule

$$(\phi_1, \chi_{g_1}, g_1)(\phi_2, \chi_{g_2}, g_2) = (\phi_1 + \psi(g_1)\phi_2 + f(g_1, g_2), \chi_{g_1 g_2}, g_1 g_2) \tag{2.6}$$

where $f(g_1, g_2)$ is given by

$$f(g_1, g_2) \overset{\text{def}}{=} \chi_{g_1}(x) + \psi(g_1)\, \chi_{g_2}(x) - \chi_{g_1 g_2}(x) \tag{2.7}$$

and is easily shown from (2.4) to be a constant $\forall g_1, g_2 \in G_F$. It follows also from the definition (2.4) that f satisfies on $G_F \times G_F$ the factor set condition

$$f(g_1,g_2) + f(g_1g_2,g_3) = f(g_1,g_2g_3) + \psi(g_1)\, f(g_2,g_3) \; . \qquad (2.8)$$

The set of automorphisms $\psi(G_F)$ in (2.6), (2.7) and (2.8) is given, using (2.2), (2.4) and the fact that ∂ transforms covariantly under the Poincaré group, by

$$\psi(g)\; \chi(x) = \varepsilon(g)\; P_g\; \chi(x) = \varepsilon(g)\; \chi(g^{-1}x) \qquad (2.9)$$

with P_g the substitution operator, and with $\varepsilon(g) = \text{sign } L^o_o$ (L the homogeneous part of g). In particular $\psi(G_F)$ is simply given by $\varepsilon(g)$ when restricted to $\mathcal{R}$. The group $\overline{Q}$ then appears as an extension of $\mathcal{R}$ by G_F as is shown in the following commutative diagram of group extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[d, tail] & \overline{Q} \arrow[r] \arrow[d, tail, "\mu"] & G_F \arrow[r] \arrow[d, tail] & 1, & f,\psi \\
0 \arrow[r] & J \arrow[r] & JP \arrow[r] & P \arrow[r] & 1, & 0,\psi
\end{tikzcd} \qquad (2.10)$$

where J is an (in general infinite dimensional and not locally compact) group of real functions over M(4), the abelian group of the allowed gauge transformations, and JP can be taken as a semi-direct product of J by P (see [14]).

As mentioned above the gauge functions χ_g are determined (for a given potential) up to a constant. It follows from proposition 3 of [13] that we may always fix these constants in such a way that μ, which defines the imbedding of $\overline{Q}$ in JP

$$\mu\;(\phi,\chi_g,g) \;\stackrel{\text{def}}{=}\; (\phi + \chi_g, g) \in JP \qquad (2.11)$$

is a monomorphism.

Actually $\overline{Q}$, as an abstract group, consists of pairs (ϕ,g), $\phi \in \mathcal{R}$, $g \in G_F$ as follows from (2.10). We have written its elements (ϕ,χ_g,g) for clarity only, making explicit the interpretation of $\overline{Q}$ as defined by μ and A.

The explicit structure of $\overline{Q}$ still may depend on the specific choice of potential we started with. However, it can also be shown that different choices for the potential give rise to equivalent extensions in (2.10) and hence to isomorphic groups. For proofs and details we refer to [13] and [14].

In fact it is sufficient, from a physical point of view, to consider only a subgroup Q of $\overline{Q}$, namely the subgroup generated by all elements of the form $\{(0,\chi_g,g),\ g \in G_F\}$. Denoting therefore by $\mathcal{F}$ the subgroup of the real line generated by the factor set f in (2.7) ($\mathcal{F} = \{f(g_i,g_j),\ \forall g_i,g_j \in G_F\}$) the diagram (2.10) reduces to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & Q & \longrightarrow & G_F & \longrightarrow & 1, \quad f,\psi \\ & & \downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & JP & \longrightarrow & P & \longrightarrow & 1, \quad 0,\psi \end{array} \tag{2.12}$$

We call Q the <u>symmetry group of the potential.</u> Since it does not, as an abstract group, depend on the choice we can make for the potential, we may restrict ourselves to some special convenient gauge and calculate explicitly the factor set f which is of course of essential importance in the determination of the structure of Q. That is what we shall do in the remaining part of this section.

Let us thus define a particular mapping π from the e.m. field tensors to the space of the potentials

$$\pi : \quad F \longmapsto A \tag{2.13}$$

as follows: noting also that for various kinds of fields there exist more natural choices we split first the spectrum S of the field in three disconnected parts:

$$S = S^{(o)} \cup S^{(r)} \cup S^{(j)}$$

$$S^{(\alpha)} \cap S^{(\beta)} = \emptyset \qquad \alpha \neq \beta,\ \alpha,\ \beta \in o,\ r,\ j$$

with

$$\begin{aligned} S^{(o)} &= \{\, k \in S \mid k = 0 \,\} \\ S^{(r)} &= \{\, k \in S \mid k^2 = 0,\ k \neq 0 \,\} \\ S^{(j)} &= \{\, k \in S \mid k^2 \neq 0 \,\} \end{aligned} \tag{2.14}$$

This decomposition is clearly Poincaré invariant. The field F splits correspondingly to (2.14) in three (independent) e.m. fields

$$F_{\mu\nu}(x) = F^{(o)}_{\mu\nu} + F^{(r)}_{\mu\nu}(x) + F^{(j)}_{\mu\nu}(x) \tag{2.15}$$

where $F^{(o)}_{\mu\nu}$ will be assumed for the sake of this work and because of its physical interest to consist of a constant uniform field only (as is always the case for a periodic field for example). Further $F^{(r)}_{\mu\nu}(x)$ is for obvious reasons called a radiation field and $F^{(j)}_{\mu\nu}(x)$ a "current" field. These different parts are clearly defined, for $\alpha = o, r, j$, by

$$F^{(\alpha)}_{\mu\nu}(x) = \int d^4k\ \hat{F}^{(\alpha)}_{\mu\nu}(k) \exp(ikx) \tag{2.16}$$

where

$$\hat{F}^{(\alpha)}_{\mu\nu}(k) = \begin{cases} \hat{F}_{\mu\nu}(k) & , \quad k \in S^{(\alpha)} \\ 0 & \text{else} \end{cases} \quad . \tag{2.16'}$$

Because each of these fields has to satisfy independently the Maxwell equations, we choose for each of them a corresponding potential, in a specific, convenient gauge. We define thus the mapping π of (2.13) as acting on the field F by

$$(\pi F)_\mu(x) \overset{\text{def}}{=} A^{(o)}_\mu(x) + A^{(r)}_\mu(x) + A^{(j)}_\mu(x) = A_\mu(x) \tag{2.17}$$

with

$$A^{(o)}_\mu(x) \overset{\text{def}}{=} \tfrac{1}{2}\, x^\rho\, F^{(o)}_{\rho\mu} \qquad \text{(symmetric gauge)}$$

$$A^{(r)}_\mu(x) \overset{\text{def}}{=} \int d^4k\, \frac{\hat{F}^{(r)}_{o\mu}(k)}{ik_o} \exp\,(ikx) \qquad \text{(radiation, or Coulomb gauge)} \tag{2.18}$$

$$A^{(j)}_\mu(x) \overset{\text{def}}{=} \int d^4k\, \frac{k^\rho\, \hat{F}^{(j)}_{\rho\mu}(k)}{k^\sigma k_\sigma} \exp\,(ikx) \qquad \text{(Lorentz gauge)} \quad .$$

The image $A = \pi F$ of the field F under π will be called the <u>standard potential</u> of F. It may be verified, using the Maxwell equations (1.1) for the Fourier coefficients of the field, that this potential indeed satisfies (2.1). Further it is now quite an easy matter to use the ansatz (2.18) to calculate the compensating gauges (2.3) and hence, the factor sets (2.7). After a long but straightforward calculation, using the invariance property of the field under $g = (a,L) \in G_F$, and the Lorentz invariance of the Lebesgue measure d^4k we obtain for the various parts of the field separately

$$\chi_g^{(o)}(x) = -\tfrac{1}{2} F_{\sigma\rho}^{(o)} t^\sigma x^\rho + c_1$$

$$\chi_g^{(r)}(x) = \int^{x^o} dx^o \, (g A^{(r)})_o(x) + c_2 \tag{2.19}$$

$$\chi_g^{(j)}(x) = c_3$$

where c_1, c_2, and c_3 are constants which can be chosen 0 so to satisfy (2.11), $t = a + u(L)$, $a \in U_F$ a pure translation symmetry in G_F, and $u(L)$ a translation associated to the Lorentz transformation L. Such a translation $u(L)$ can also be called <u>non-primitive</u>, in analogy with the usual nomenclature of space groups. For the use of these crystallographic concepts in the frame of subgroups of the Poincaré group, see [8].

The factor sets (2.7) are then obtained from (2.19) as given by

$$f^{(o)}(g_1,g_2) = \tfrac{1}{2} F_{\sigma\rho}^{(o)} (L_1 t_2)^\sigma (t_1)^\rho$$

$$f^{(r)}(g_1,g_2) = f^{(j)}(g_1,g_2) = 0 \quad . \tag{2.20}$$

Using the linearity of Poincaré transformations and of our map π in (2.17) and (2.18) on potentials and fields respectively we obtain the corresponding compensating gauge and factor set for the whole field by simple addition of the various parts so that finally we have

$$\text{(i)} \quad \chi_g(x) = -\tfrac{1}{2} F_{\sigma\rho}^{(o)} t^\sigma x^\rho + \int^{x^o} dx^o \, (gA^{(r)})_o(x)$$

$$\text{(ii)} \quad f(g_1,g_2) = \tfrac{1}{2} F_{\sigma\rho}^{(o)} (L_1 t_2)^\sigma t_1^\rho \quad . \tag{2.21}$$

Let us note that if G_F is a symmorphic space-time group (i.e. if all $u(L)$, $\forall L \in K_F$, can be chosen 0), or if the set of all $u(L)$ can be chosen in the real vector space spanned by U_F, the factor set (2.21) (ii) is equivalent to zero if and only if its restriction to $U_F \times U_F$ vanishes identically. This follows from the fact that the factor set in (2.21) (ii) is bilinear in the translations and that a trivial factor set is necessarily symmetrical when restricted to the translation subgroup whereas in (2.21) (ii) $f(a_2,a_1) = - f(a_1,a_2)$, $\forall\ a_1,a_2 \in U_F$, i.e. is totally antisymmetric.

The results (2.21) are quite remarkable, seen the large class of fields we are considering: it follows indeed from (2.21) (ii) that the upper extension in (2.12) is trivial unless the field has a Fourier component at the origin whose contribution to the field is different from zero, i.e. for the class of fields we consider, unless the field carries a constant uniform part. As a consequence invariance operator groups will be homomorphic to a subgroup of the covering group of the Poincaré group up to a factor system which may be non-trivial when and only when this contribution is different from zero.

As a result we have now found the explicit structure of the potential symmetry groups. These are however not yet the most relevant ones for our purposes and that is why it is not the structure of these symmetry groups that we are going to analyze further but that of the corresponding invariance operator groups we can now define from them and which are constructed in such a way that they commute with the operators of the appropriate equations of motion.

3. Relativistic invariance operator groups, discrete and continuous parts.

Let us now consider a charged, relativistic, massive particle of spin 0 or $\frac{1}{2}$ moving in a given external e.m. field, and thus obeying the Klein-Gordon, respectively the Dirac equation with minimal coupling. We define, from the symmetry group Q of the corresponding potential, an operator group $\hat{\mathcal{Q}}_s$, by the following map Φ_s (s the spin)

$$\Phi_s : Q \longrightarrow U(\mathcal{H})$$

with $U(\mathcal{H})$ the group of unitary (anti-unitary) operators in the separable carrier Hilbert space $\mathcal{H}$ of physical states and, for $(f,\chi_g,g) \in Q$, $f \in \mathcal{F}$, $g = (a,L) \in G_F$

$$\Phi_s((f,\chi_g,g)) \overset{\text{def}}{=} \exp\{-i\frac{e}{c\hbar}(f+\chi_g(x))\}\, S(L).P_g \overset{\text{def}}{=} \langle f,g\rangle_s \quad (3.1)$$

where P_g is the substitution operator, e is the charge of the particle, c the speed of light and $\hbar$ the Planck constant divided by 2π. Furthermore S(L) is given by

$$\text{(i)}\quad S(L) = \begin{cases} C & , L^o_c < 0 \\ 1 & , L^o_o > 0 \end{cases}, \quad \text{for } s = 0 \quad \text{(Klein-Gordon)}$$

$$(3.2)$$

$$\text{(ii)}\quad S(L) = \begin{cases} S_o(L)\cdot C & , L^o_o < 0 \\ S_o(L) & , L^o_o > 0 \end{cases}, \quad \text{for } s = \tfrac{1}{2} \quad \text{(Dirac)}$$

where C is the charge conjugation operator and $S_o(L)$ satisfies the following conditions on the γ-matrices

$$S_o(L)^{-1}\,\gamma^\mu\, S_o(L) = L^\mu_\nu\,\gamma^\nu \quad . \quad (3.3)$$

The images $\mathcal{Q}_s$ of Q under Φ_s are called the invariance operator groups as they can be shown to leave the corresponding equation of motion invariant [13], i.e. to commute, for s = 0 and s = 1/2 with the operators

$$O_{KG}(x,p,\pi F) \equiv (p - \frac{e}{c}(\pi F)(x))^2 - m^2c^2 \quad , \quad s = 0$$

and respectively

$$O_D(x,p,\pi F) \equiv -i\gamma^\mu (p_\mu - \frac{e}{c}(\pi F)_\mu(x)) - mc \ , \quad s = 1/2$$

where $\pi F = A$ is the potential defined in (2.17) and (2.18).

This means that the invariance operator group of the equation of motion is (up to the usual spin factor system) homomorphic to the symmetry group of the potential (of the field considered): indeed, up to the factor system that may be generated by the matrices S(L) Φ_s is an homomorphic mapping on Q, as is easily seen.

As a general result we have thus found now the explicit structure of the invariance operator groups for (almost) arbitrary external e.m. field. As these groups leave, by definition, the corresponding equations of motion invariant, the solutions of these equations will span representation spaces for these groups and the problem of the characterisation and of the properties of the solutions can thus successfully be tackled by the study of the corresponding representations. These representations can in principle be found by standard group theoretical methods, provided certain regularity conditions are satisfied. Unfortunately there are also cases where these conditions are not satisfied and it is this problem that we shall discuss now.

Let us therefore use the splitting of G_F into connected and discrete parts as in (1.6). Whenever the upper exact sequence is split there exists a monomorphic section $r : G_F^d \longrightarrow G_F$, and hence G_F^d may be identified with a discrete subgroup of G_F. Let us now consider for this case the corresponding operator groups $\mathcal{Q}_s^c = \Phi_s(Q^c)$ and $\mathcal{Q}_s^d = \Phi_s(Q^d)$ first separately, with Q^c and Q^d defined analogously as in (2.12), G_F being replaced by its subgroups G_F^c or G_F^d respectively. Note that, since the choice of an (even monomorphic) section r as above is not necessarily unique, the structure of $\mathcal{Q}_s^d$ may depend on this choice. This apparent arbitrariness will, of course, be removed when we shall consider again the whole invariance operator group $\mathcal{Q}_s$, in the next section.

We come now to the following

Proposition 3.1. Let $|K_F^d| < \infty$, then there exists a (monomorphic) section r such that $\mathcal{Q}_s^d$ is of Type I if and only if the field is rational (in a sense explained below).

Before we go to the proof let us explain in more detail the contents of this proposition. First a group is said to be of Type I (or equivalently to be tame) if its representations (continuous unitary in a separable Hilbert space) generate von Neumann algebras having only discrete (as opposed to continuous) factors. Otherwise it is termed non-Type I (or wild).

We recall that a factor is a von Neumann algebra which has a trivial centre, and it is called discrete if it is isomorphic to a von Neumann algebra whose commutant is abelian [15]. In particular all abelian, finite or compact groups, all connected semisimple or nilpotent Lie groups and all connected real algebraic groups are of Type I.

Non-Type I groups are, so to say, a mathematical pathological curiosity and only recently a simple quantum mechanical system giving rise to such an invariance group has been discovered [6]. An important feature of non-Type I groups is that there exists no mathematical theory (and not much hope for such a one) which would allow us, even in principle, to derive or describe all their irreducible (unitary) representations. Moreover, given a representation, although the decomposition into primaries (representations generating von Neumann algebras which are factors) is unique (the so-called central decomposition) the decomposition of a primary into irreducible constituents can be fantastically non-unique and this leads of course to serious difficulties, in particular with the physical interpretation of the irreducible representations. For further details on the mathematical aspects of these pathological groups we refer to the literature (see e.g. [11],[15-19]) and for a short and simple description to the presentation by Boon [6].

We then define a field to be <u>rational</u> if and only if there exists a (monomorphic) section r: $G_F^d \dashrightarrow G_F$ such that the set $\{\Phi_s(f(g_i,g_j),0,1), \forall g_i,g_j \in r(G_F^d)\}$ is finite. For $|K_F^d| < \infty$, this is equivalent to the following: let $\{b_1,\ldots,b_{n_o}\}$ be a basis of a n_o-dimensional discrete translation subgroup as defined by (1.7) and the section r, then the field is rational if and only if $\{b_1,\ldots,b_{n_o}\}$ can be chosen in such a way that the factor system satisfies the condition

$$\frac{1}{2\pi}\frac{e}{ch} f(b_i,b_j) \in \mathbb{Q} \quad , \qquad \forall i,\ j = 1,\ 2,\ \ldots\ n_o \qquad (3.4)$$

i.e. using (2.21)

$$\frac{1}{(ch/2e)} F^{(o)}_{\rho\mu}(b_i)^\rho(b_j)^\mu \in \mathbb{Q}, \qquad \forall i,\ j = 1,\ \ldots\ n_o\ . \quad (3.5)$$

The equivalence follows from the facts that f is bilinear, when restricted to the translations, that $L \in K_F$ has, in this basis, only integer entries and hence, since $|L| < \infty$, a non-primitive translation associated to L can always be chosen with rational components with respect to $\{b_1,\ldots,b_{n_o}\}$ (a linear system of equations with rational coefficients having always a rational solution).

We now turn to the proof of the proposition: it follows in fact from the theorem by Thoma [11] mentioned in section 1, and is further analogous to the proof by Boon [6], that it contains as a particular case (time independent problem of a Bloch electron in a c.u. magnetic field [1]): indeed $A \lhd \mathcal{L}_s^d$ implies A normal in the subgroup of $\mathcal{L}_s^d$ generated by discrete translations only, and the condition that A is then abelian and of finite index is easily shown to be just the rationality condition (3.5). Since $|K_f^d| < \infty$, A is then also of finiteindex in the whole $\mathcal{L}_s^d$ and this completes the proof.

In the cases where $|K^d| = \infty$, it is also possible to show that the operator group $\mathcal{L}_s^d$ will only be of Type I if G^d is itself of Type I (for the latter, see Proposition 1.4) and if the field is rational as above. As explained in section 1 we do not however go further into detail for this case. Finally it may also be mentioned here that $\mathcal{L}_s^c$, the operator group generated by the connected component of G_F, is always of Type I.

We think it useful at this point to make a few additional comments on the content and consequences of the above Proposition 3.1.:

(i) We note that the expression in the denominator of (3.5) is equal to the magnetic flux quantized unit, so that the expression (3.5) becomes, for the case of a Bloch electron in a c.u. magnetic field, equal to the magnetic flux (in quantized units) through the wall spanned by the (space) vectors $\vec{b}_i$ and $\vec{b}_j$ [5,6]. This may give the idea that the effect is of pure magnetic nature, but this is contradicted by the following:

(ii) Let us mention a few examples of simple physical systems giving rise to these pathological invariance operator groups. Consider a crystal in a condensor with, in addition, a pure time-dependent periodic electric field: the expression in (3.5) is then different

1) Note that in [6], only the non-relativistic system is considered, and only translational symmetries, but this does not imply any essential change in the proof.

from zero and the invariance operator group will vary wildly with very small changes in the strength of the condensor field, for example. Mixed electric and/or magnetic effects may be obtained when for instance, a charged particle moves in the superposition of two (non parallel) plane waves (as for crossing lasers), or in wave guides. Adding to this system a constant uniform electric and/or magnetic field will again produce pathological invariance operator groups.

(iii) It may happen that $\mathcal{Q}_s^d$ is not of Type I whereas $\mathcal{Q}_s$ itself is. We shall see in the next section more in detail how this can occur and how sensitive the situation then becomes.

(iv) We note that, as soon as one term in (3.5) is different from zero, Q^d, the discrete part of the symmetry group of the potential, is never of Type I, but this is in our problem not relevant, the mapping Φ_s making just restrictions on the physical representations (the constant $e/c\hbar$ being in fact not arbitrary).

(v) It has also been clearly shown from the example in [6] (see also [20]) that if one imposes periodic boundary conditions, allowing thus only "rational" values to the field, the dimensions of the irreducible representations may change very discontinuously with an infinitesimal (rational) change of the values of the field, and in fact this will be the case for any "rational" field for which the expression in (3.5) does not vanish identically. Such a condition does not therefore get us out of the difficulties and in any case its meaning is in general physically not obvious.

We now turn to a more precise description of the structure of the whole invariance operator group, using the results for its various parts as obtained so far.

4. The whole invariance operator group $\mathcal{Q}_s$.

It follows directly from (3.1) that $\mathcal{Q}_s$ appears as an extension of $U(\mathcal{F})$ by $(G_F)_s$ where $U(\mathcal{F})$ is the subset of the unit circle which is the image of ϕ_s as restricted to $\mathcal{F}$, and $(G_F)_s$ is given by G_F for s integer (s = 0) and by the corresponding subgroup of the covering group of the Poincaré group for s halfinteger ($s = \frac{1}{2}$). Using now (3.1) and the decomposition (1.6) of $(G_F)_s$ into connected and discrete parts, we may write an element $\underline{q}_s \in \mathcal{Q}_s$ as

$$\underline{q}_s = \langle f,g\rangle_s = \langle f,g^c,g^d\rangle_s \tag{4.1}$$

with $g \in (G_F)_s$, g^c its connected, g^d its discrete parts respectively. The relationship between $\mathcal{Q}_s$ and the connected respectively discrete operator groups $\mathcal{Q}_s^c$ and $\mathcal{Q}_s^d$ defined in the previous section is now given by the following:

<u>Proposition 4.1.</u> Let $f(G,G^c)$ denote the subset of $\mathcal{F}$ generated by the restriction of the factor system f of (2.12) to $G \times G^c$. Then the following sequences of groups are exact:

(i) if $f(G,G^c) = 0$, then $(G_F^c)_s \cong {}^c_s$ and

$$1 \longrightarrow (G_F^c)_s \longrightarrow \mathcal{Q}_s \longrightarrow \mathcal{Q}_s^d \longrightarrow 1 \tag{4.2}$$

(ii) if $f(G,G^c) \neq 0$, then

$$1 \longrightarrow \mathcal{Q}_{so} \longrightarrow \mathcal{Q}_s \longrightarrow (G_F^d)_s \longrightarrow 1 \tag{4.3}$$

where $\mathcal{Q}_{so}$ is the subgroup of $\mathcal{Q}_s$ generated by $\mathcal{F}$ and $(G_F^c)_s$.

The proof of this Proposition is straightforward, using the structure of the group $\mathcal{Q}_s$ as obtained so far and the decomposition (4.1). It is thus omitted.

The interest of this proposition lies in that it allows us now to give a first answer to the question whether or not the Type I respectively non-Type I property of $\mathcal{L}_s^d$ does carry over the whole of $\mathcal{L}_s$. Indeed

<u>Proposition 4.2.</u> In case (i) of the preceding proposition, if $\mathcal{L}_s^d$ is not of Type I then $\mathcal{L}_s$ is also not of Type I.

<u>Proof:</u> it follows straightforwardly from theorem (8.1) of [19] that if a group has a normal subgroup such that the corresponding factor group is not of Type I, then it is itself not of Type I. The proposition is then implied by the exactness of the sequence (4.2).

The remarkable fact is now that this proposition does not in general hold in case (ii) and the situation may be very critical. It is of course possible to give some complicated criteria to characterize this situation but we think it preferable to show what may then happen with a simple representative example.

5. Example: a Bloch electron in a c.u. e.m. field.

Let us consider a Bloch electron in a c.u. electromagnetic field $F_{\mu\nu}^{(o)}$, and translational field symmetries in space and time only so that $G_F = U_F \cong \mathcal{R} \oplus \mathbb{Z}^3$, $\mathcal{R}$ standing for time translations. A set of generators for U_F is given by $\{\lambda e^{(o)} = \lambda(1,0,0,0),\ \lambda \in \mathcal{R}$ and a base $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of the crystal (space) lattice$\}$. We assume further for simplicity that $\vec{b}_3$ is along the z-axis and $\vec{b}_1$ and $\vec{b}_2$ are in the (x,y)-plane. Let us then first consider the situation where $F_{oi}^{(o)} = E_i = 0$, $F_{ij}^{(o)} = \varepsilon_{ijk} B_k$ with $\vec{B} = (0,0,B_3)$. This is the situation of [6] and in our language, it is characterized by the following factor set of (2.12) as directly obtained from (2.21)

$$f(\vec{b}_i,\vec{b}_j) = \tfrac{1}{2}\, B_3 \left[(\vec{b}_i)_1(\vec{b}_j)_2 - (\vec{b}_j)_2(\vec{b}_i)_1 \right] = \tfrac{1}{2}\, (\vec{B},\vec{b}_i \wedge \vec{b}_j)$$

$$f(\vec{b}_i,\lambda e^{(o)}) = f(\lambda e^{(o)},\lambda' e^{(o)}) = 0 \tag{5.1}$$

$\mathcal{F}$ is then a discrete subgroup of $\mathcal{R}$ and we are in case (i) of the Proposition 4.1. The situation may be characterized by the following commutative diagram of exact sequences

$$\begin{array}{ccccc} & & \mathcal{Q}_s^c & \rightarrowtail & \mathcal{R} \\ & & \downarrow & & \downarrow \\ U(\mathcal{F}) & \rightarrowtail & \mathcal{Q}_s & \twoheadrightarrow & \mathcal{R} \oplus \mathbb{Z}^3 \\ \downarrow & & \downarrow & & \downarrow \\ U(\mathcal{F}^d) & \rightarrowtail & \mathcal{Q}_s^d & \twoheadrightarrow & \mathbb{Z}^3 \end{array} \tag{5.2}$$

where, as is easily verified, the central column is split (this is not always so, as we shall see in a moment, even if G_F is a direct product of G_F^c by G_F^d as it is in our example). This case is of course the one discussed in [6] and by Proposition 4.2 (and the fact that the central extension is regular in the sense of Mackey [19]), $\mathcal{Q}_s$ is of Type I if and only if $\mathcal{Q}_s^d$ is.

Let us now also introduce an <u>electric</u> c.u. field, ($F_{oi}^{(o)} \neq 0$) with $\vec{B}$ remaining parallel to $\vec{b}_3$ along the z-axis and $\vec{b}_1$, $\vec{b}_2$ in the (x,y)-plane. We now have, from (2.21):

$$f(\vec{b}_i,\vec{b}_j) = \tfrac{1}{2}\, (\vec{B},\vec{b}_i \wedge \vec{b}_j)$$

$$f(\vec{b}_i,\lambda e^{(o)}) = \tfrac{1}{2}\, \lambda (\vec{E},\vec{b}_i) \neq 0 \qquad (\lambda \in \mathcal{R}) \tag{5.3}$$

$$f(\lambda e^{(o)},\lambda' e^{(o)}) = 0$$

so that we are in case (ii). The situation is now illustrated, in place of (5.2), by the commutative diagram

$$
\begin{tikzcd}
U(1) \arrow[r, tail] \arrow[d, dashed] & \mathcal{L}_{so} \arrow[r, dashed, two heads] \arrow[d, tail, "(\iota)"] & \mathbb{R} \arrow[d, tail] \\
U(\mathcal{F}) \arrow[r, tail] & \mathcal{L}_s \arrow[r, two heads] \arrow[d, two heads] & \mathbb{R} \oplus \mathbb{Z}^3 \arrow[d, dashed, two heads] \\
 & \mathcal{L}_s/\mathcal{L}_{so} \arrow[r, dashed, two heads] & \mathbb{Z}^3
\end{tikzcd}
\tag{5.4}
$$

The middle column extension of (5.4) is now characterized by a (non-trivial) factor system m and a mapping ϕ, which can be straightforwardly computed, using our preceding results: The factor system m is given (after identification of $\mathcal{L}_s/\mathcal{L}_{so}$ with $U_F^d \equiv \mathbb{Z}^3$ and imbedding in $\mathcal{L}_s$) by

$$
(\iota)\ m(\vec{b}_i,\vec{b}_j) = \langle f(\vec{b}_i,\vec{b}_j), C\rangle_s \tag{5.5}
$$

with $\langle f,g\rangle_s$ as in (3.1). The homomorphic map $\phi: \mathbb{Z}^3 \longrightarrow \mathrm{Aut}\,\mathcal{L}_{so}$ is then obtained, using the antisymmetry and linearity properties of the factor system f on translations, as explicitly given by

$$
\phi(\vec{b}) \langle f,\lambda e^{(o)}\rangle_s = \langle f + 2\, f(\vec{b},\lambda e^{(o)}),\lambda e^{(o)}\rangle_s \tag{5.6}
$$

where $\vec{b} = \Sigma_i\, n_i\, \vec{b}_i \in U_F^d \cong \mathbb{Z}^3$.

In order to determine if $\mathcal{L}_s$ is of Type I, we have to analyse in some more detail its representations which we shall now calculate, by induction, from the irreducible representations of $\mathcal{L}_{so}$, using the general theory of Mackey [19]. The irreducible representations of $\mathcal{L}_{so}$ are, since $\mathcal{L}_{so}$ is abelian and isomorphic to the direct product $U(1) \otimes \mathbb{R}$, directly given by the characters $\{\hat{q}_o^{n,\varepsilon}\}$ with $n \in \mathbb{Z}$, $\varepsilon \in \mathbb{R}$ and

$$\hat{q}_o^{n,\varepsilon} < f, \lambda e^{(o)} >_s = \exp(i\, n \frac{e}{c\hbar}\; f) \exp(i \varepsilon\, \lambda) \tag{5.7}$$

The physical representations defined by (3.1) will then correspond to $n = -1$ and ε stands for the energy (in reduced dimensionless units). The dual $\hat{\mathcal{L}}_{so}$ thus consists of a countably infinite set of parallel lines in a 2-dimensional vector space and is itself of course of Type I. The action $\hat{\phi}$ of $\mathbb{Z}^3$ on this dual is then given by

$$(\hat{\phi}(\vec{b})\; \hat{q}_o^{n,\varepsilon}) < f, \lambda e^{(o)} >_s \stackrel{\text{def}}{=} \hat{q}_o^{n,\varepsilon}(\phi(\vec{b})^{-1} < f, \lambda e^{(o)} >_s) \tag{5.8}$$

so that, using (5.3) and (5.6), we find

$$\hat{\phi}(\vec{b})\; \hat{q}_o^{n,\varepsilon} \;=\; \hat{q}_o^{n,\varepsilon} - \frac{ne}{c\hbar} (\vec{E}, \vec{b}) \tag{5.9}$$

This means that the stability group (or little group) of $\hat{q}_o^{n,\varepsilon}$ is the whole of $\mathcal{L}_s$ for $n = 0$, and for $n \neq 0$ the extension of $\mathcal{L}_{so}$ by the subgroup A of $\mathbb{Z}^3$ which corresponds to all lattice translations $\vec{a}$ satisfying the condition $(\vec{E}, \vec{a}) = 0$. In the case $n = 0$, the induced representations are all 1-dimensional and of course of Type I. We may thus omit in the sequel this (unphysical) case. For $n \neq 0$ we now induce the representations $\hat{q}_o^{n,\varepsilon}$ to $\mathcal{L}_s$ by the following procedure: consider an element $\hat{q}_o^{n,\varepsilon}$ as in (5.7) and its orbit $O_{n,\varepsilon}$ (i.e. the set of its images under the action (5.9) of $\hat{\phi}(\mathbb{Z}^3)$). Let now μ be the corresponding unique (as a class) measure which is invariant, ergodic and transitive (under this action). Then comes the first difficulty: it follows from (5.9) that all ergodic measures will be transitive (i.e. concentrated on one orbit) if and only if, $\forall\, i, j$ such that $(\vec{E}, \vec{a}_i)$ and $(\vec{E}, \vec{a}_j)$ are non-zero, one has $(\vec{E}, \vec{a}_i) / (\vec{E}, \vec{a}_j) \in \mathbb{Q}$ (these are then the atomic measures). We assume in a first step that this condition is fulfilled. Further, since they are in one-to-one Borel correspondence we identify, as is usual, the orbit, the coset space and the coset representatives (which we denote $\{a_i\}$).

Let Γ be a projective representation of A with some factor set ω (to be specified later on) and $\mathcal{L}(0_{n,\varepsilon},\mu)$ denote the (separable) Hilbert space of functions ψ from $0_{n,\varepsilon}$ to the carrier space $\mathcal{H}$ of $\hat{q}_o^{n,\varepsilon} \otimes \Gamma$, satisfying the two conditions

(i) $(\phi, \psi(\vec{a}_i))$ is μ-measurable $\forall \phi \in \mathcal{H}$

(ii) $\int_{0_{n,\varepsilon}} \| \psi(\vec{a}_i) \|^2 \, d\mu < \infty$ (with the norm taken in $\mathcal{H}$).

The induced represenation of $\mathcal{L}_s$ (from $\hat{q}_o^{n,\varepsilon}$, by Γ) is then defined by

$$(\hat{q}_o^{n,\varepsilon}\uparrow \mathcal{L}_s)^{\Gamma} < f,\lambda e^{(o)}+ \vec{b}>_s \psi(\vec{a}_i)$$

$$\stackrel{\text{def}}{=} (\hat{q}_o^{n,\varepsilon}\cdot \Gamma) (<o,\vec{a}_i>_s \; <f,\lambda e^{(o)}+ \vec{b}>_s \; <o,\vec{a}_k>_s^{-1}) \psi(\vec{a}_k) \tag{5.10}$$

$$\equiv (\hat{q}_o^{n,\varepsilon}\cdot \Gamma) (<f_1,\lambda e^{(o)}+ \vec{a}>_s) \psi(\vec{a}_k)$$

$$\stackrel{\text{def}}{=} (\hat{q}_o^{n,\varepsilon} (<f_1,\lambda e^{(o)}>_s) \otimes \Gamma(\vec{a})) \psi(\vec{a}_k)$$

where $\vec{a}_k$ is the (unique) coset representative satisfying the condition $\vec{a}_i + \vec{b} - \vec{a}_k = \vec{a} \in A$; further, f_1 is obtained from the product rule of $\mathcal{L}_s$ to be equal to

$$f_1 = f + f(\vec{a}_i + \vec{a}_k, \lambda e^{(o)}+ \vec{b}) + f(\vec{a}_k,\vec{a}_i) \quad .$$

The factor set ω is, as is well known, uniquely (as a class) defined by $\hat{q}_o^{n,\varepsilon}$ and the action $\hat{\phi}$. Since $\mathcal{L}_{so}$ is abelian, ω is simply given (for $\vec{a}$, $\vec{a}' \in A$), by

$$\omega(\vec{a},\vec{a}') = \hat{q}_o^{n,\varepsilon} <f(\vec{a},\vec{a}'), 0>_s \quad ; \tag{5.11}$$

so that Γ is, by the lifting procedure, an ordinary representation of a subgroup of the operator group $\mathcal{L}_s^d$ analyzed in the first part of our example (the operator group induced by the discrete space translations only). If and only if this subgroup is of Type I for all $\hat{q}_o$ (we still assume that the ergodic measure μ is also transitive) it is true, again from theorem 8.1 of [19], that $\mathcal{L}_s$ will be of Type I as well. It is now possible to describe how sensitive the situation is. We have in fact the following possibilities:

a) $\vec{E} \parallel \vec{B}$ ($\parallel$ z-axis $\parallel$ $\vec{b}_3$), then A is generated by all lattice translations in the (x,y)-plane, and the situations is as in [6]: $\mathcal{L}_s$ is of Type I if and only if $\mathcal{L}_s^d$ is, i.e. if and only if the magnetic flux is rational through any lattice wall.

b) $\vec{E} \perp \vec{B}$ and suppose there exists some lattice vector $\vec{a} \nparallel \vec{b}_3$ such that $(\vec{E}, \vec{a}) = 0$. Together with $\vec{b}_3$, A consists then of two dimensional translations. The factor set ω in (5.11) is however trivial, as follows from (2.21) and Γ is thus a one-dimensional ordinary representation of A. The induced irreducible representations are then of Type I, as also the action of $U_F^d/A \cong \mathbb{Z}$ is easily shown to be regular. Hence $\mathcal{L}_s$ is then of Type I, and for any value of $|\vec{B}|$. This means that by introducing an (even arbitrarily small) c.u. electric field in such a direction in the example of [6], all non-Type I pathology can be removed (whereas the space-time symmetry of the physical system has in fact remained the same), and this is quite remarkable.

c) $\vec{E} \nparallel \vec{B}$, and even if there exists a translation $\vec{a}$ as in the previous case, it may happen that for two other linearly independent lattice vectors $\vec{b}$ and $\vec{b}'$ we have $(\vec{E}, \vec{b}) \neq 0 \neq (\vec{E}, \vec{b}')$ with an irrational ratio: $(\vec{E}, \vec{b}) / (\vec{E}, \vec{b}') \notin \mathbb{Q}$. Then it follows from (5.9) that the Lebesgue measure on each real line with fixed n is ergodic but not transitive, i.e. strictly ergodic: This follows from the fact that each orbit, being a countable set, has measure 0 (see e.g. [18]). As a consequence, the extension $\mathcal{L}_s$ of $\mathcal{L}_{so}$ by $\mathbb{Z}^3$ is non-regular, in the sense of Mackey [19] and the whole theory breaks down again.

It is possible to show (using for example Thoma's theorem) that here again $\mathcal{L}_s$ is not of Type I. This is a new kind of pathology since the effect does not depend on the <u>absolute value</u> of the field $\vec{E}$ (as for $\vec{B}$ previously) but <u>only</u> on its <u>direction</u> with respect to the lattice.

We hope we have shown quite clearly with this example how critical the situation has become. These results may look physically quite unrealistic but we think they may be helpful: indeed, a major problem with the occurence of such groups in simple physical systems is, besides the conceptual problems related to the interpretation of group theory in quantum theory (e.g. the role of the irreducible representations), the question of whether or not physical properties (such as selection rules, observables, conserved quantities and so on) are affected by this pathological behaviour. In fact this question is up to now far from being clear, even in the previously known case [6]. In the above example b), one can for instance take the limit of $\vec{E} \to 0$ and the groups remain of Type I, $\forall\, \vec{E} \neq 0$, but may converge in the limit to a group that is not of Type I. One could in this way study much better the possible consequences of this occurrence of non-Type I groups in such physical systems. Taking the limit of rational B-fields converging on $\mathbb{R}$ to an irrational value does not offer such a possibility because of the very discontinuous changes that occur in the dimensions of the representations. Secondly, the large class of physical systems for which the criterion (3.5) becomes critical shows also that the problem of a Bloch electron in a c.u. magnetic field is not an isolated curiosity and offers also other possibilities for analyzing the problems that arise.

6. Generalization to the Schrödinger equation.

Up to now we have considered the relativistic case, and thus Poincaré transformations only. Since we can do so with but little extra effort, we now briefly resume the very analogous situation for Galilean transformations and for the Schrödinger equation:

Proposition 6.1. If we replace the Poincaré by the Galilei group, whose action is defined on fields and potentials, for $g = (t,\Lambda)$, t a 4-translation, $\Lambda = \begin{pmatrix} \varepsilon & 0 \\ \vec{v}/c & \alpha \end{pmatrix}$ an homogeneous Galilei transformation (with $\alpha \in O(3)$, and $\varepsilon = \pm 1$), by

$$\begin{aligned} (g\vec{E})(x) &= \alpha\vec{E}(g^{-1}x) - (\frac{\vec{v}}{c} \wedge \alpha\vec{B}(g^{-1}x)) \\ (g\vec{B})(x) &= \varepsilon \ (\det \alpha) \ \ \alpha\vec{B}(g^{-1}x) \end{aligned} \tag{6.1}$$

and for ϕ the scalar, $\vec{A}$ the vector potential)

$$\begin{aligned} (g\phi)(x) &= \phi(g^{-1}x) + \frac{\vec{v}}{c} \cdot \alpha\vec{A}(g^{-1}x) \\ (g\vec{A})(x) &= \varepsilon\alpha\vec{A}(g^{-1}x) \end{aligned} \tag{6.2}$$

then all previous results remain valid, up to the obvious changes. In particular the spin factor set in (3.1) has to be replaced by the well known factor set ω_b belonging to the Schrödinger equation and expressing the fact that this equation characterizes a true projective representation of the Galilei group. This factor set is given by (see e.g. [13])

$$\omega_b(g_1,g_2) = \exp\text{-}im \left[\frac{\vec{v}_1^{\,2}}{2} - \frac{1}{2c} (\vec{v}_1+\varepsilon_2\alpha_1\vec{v}_2)^2 t_1^o + \varepsilon_1\varepsilon_2(\alpha_1\vec{v}_2,\vec{t}_1) \right] \tag{6.3}$$

More detailed results about this Galilean case, especially as concerning the validity of Galilean (non-relativistic) symmetries (in an exact or in an approximate sense) and for the problem of a charged particle in interaction with an external e.m. field, will be published separately [1]. We just reformulate here, because of their importance, the following two more precise results.

Proposition 6.2. The factor system f which determines the structure of the group of operators $\mathcal{L}_b$ commuting with the Schrödinger operator (and based on the Galilean symmetry group G_{EB} of the field $(\vec{E}, \vec{B})(x)$) may be non-trivial if and only if the Fourier transform of the field at the origin of the dual space gives a non-zero contribution. For the case where this contribution consists only in a constant uniform field $(\vec{E}^{(o)}, \vec{B}^{(o)})$ (defined similarly as in section 2) this factor system is given explicitely, for $g_i = (t_i, L(\varepsilon_i, \vec{v}_i, \alpha_i)) \in G_{EB}$, $t_i = (t_i^o, \vec{t}_i)$, $i = 1,2$, by

$$f(g_1, g_2) = \tfrac{1}{2}(\vec{E}^{(o)}, \vec{t}_2)t_1^o + \tfrac{1}{2}(\vec{B}^{(o)} \wedge \alpha_1 \vec{t}_2 - \alpha_1 \vec{E}^{(o)} t_2^o, \vec{t}_1 - \frac{\vec{v}_1}{c} t_1^o). \quad (6.4)$$

Proposition 6.3. Let $\mathcal{L}_b^d$ be defined similarly as in section 3 and K_{EB}^d correspondingly of finite order. This group is of Type I if and only if the field is rational in the same sense as in Proposition 3.1., with now as rationality condition:

$$\frac{(\vec{E}^{(o)}, \vec{b}_j)\, b_i^o - (\vec{E}^{(o)}, \vec{b}_i)\, b_j^o + (\vec{B}^{(o)}, \vec{b}_j \wedge \vec{b}_i)}{ch/2e} \in \mathbb{Q} \qquad (6.5)$$

$$\forall\, i, j = 1, \ldots, n_o$$

where $b_i = (b_i^o, \vec{b}_i)$, $b_j = (b_j^o, \vec{b}_j)$ run over the members of a basis for the corresponding space-time discrete translational symmetry group.

---

1) See also Chapter 2 of the present treatise. It is however perhaps worthwile noting here that the results of this section remain obviously valid if Euclidean transformations (with time-translations and time-reversal) only are admitted as symmetries for the external field.

Concluding remarks

Introducing a potential in the equation of motion of a free particle by means of the so-called minimal coupling is not a trivial procedure. In this chapter we have examined the consequences of this coupling by making a group theoretical analysis of an elementary quantum mechanical system in interaction with an external e.m. field. An important feature is that we may then have to do with non-Type I groups whose physical interpretation give rise to important basic problems. Let us mention with respect to this the remark of Boon [6] suggesting that the role usually attributed to irreducible representations should be perhaps given to the primaries, whose definition and determination is unique. That these non-Type I groups are not very exceptional has been shown in the generality of the criteria we have given. It is also quite remarkable that these pathological situations may occur at three distinct levels, depending on the symmetry group of the field itself, the strength of the field and its relative orientation with respect to the lattice of translations in space and time, as we saw in the examples of section 5.

More important than these considerations about the occurence of groups of non-Type I is of course the fact that we have found the explicit form of the relevant invariance operator groups (commuting with the equation of motion considered) for a charged massive particle in an (almost) arbitrary external e.m. field. It is well known that a potential has in general less space-time symmetry than the corresponding field. We have shown in fact for which kind of fields this is necessarily the case, and how we can in general restore the physical information given by the symmetries of the field. The explicit form of the invariance operator groups we have constructed is, needless to say, of fundamental importance for the treatment of a given physical system, because of the close relation between the irreducible (unitary/anti-unitary) representations of these groups and the solutions of the equations of motion and their properties.

It is perhaps worthwhile noting here that, although these invariance operator groups may of course be seen as realizations of projective representations of the space-time symmetry groups of the fields considered, this identification is in a way superficial and in fact less informative than our approach. Indeed the corresponding factor systems are for example not arbitrary on these symmetry groups as we have seen but we are uniquely determined by the fields themselves. Furthermore, even when these factor systems are known, the explicit form of the invariance operator groups still depends on the compensating gauges which are also not arbitrary, even if they generate the correct factor system for example the gauge functions in (2.19) which correspond to a radiation field cannot in general be chosen equal to zero (see e.g. [4]), whereas they always generate a trivial factor system, as we have shown.

Another important consequence of this minimal coupling prescription is then also that the relevant invariance operator groups are not, in general, subgroups of the space-time operator groups corresponding to the equation of motion considered. This implies, since invariance (which is related to the symmetry of the physical system) is necessarily a particular case of covariance (which is related to the symmetry of the physical law), that it might be interesting to analyse more in detail the concept and explicit form of covariance and covariance group in the presence of an external e.m. field. We have in fact shown elsewhere [21] that it is possible to obtain such a covariance group independently of any equation of motion on the basis of some simple physical assumptions by a generalization of the methods used in part 2 of the present chapter. Analogously as in the free case [1] it is then possible to relate the concept of covariant equations of motion to representations of this group. It follows also in particular from this treatment that for free particles of higher spins ($s \geq 1$), the introduction of the external field by simple minimal coupling is (surprisingly perhaps) not necessarily consistent.

We refer to these papers for more details. All we wish to do here is to underline some problems and emphasize new features raised by our group theoretical approach, even for well known and apparently innocent physical problems.

References

[1] E.P. Wigner, Ann. Math. 40, 149 (39)
[2] see e.g. H. Bacry et al., Nuovo Cim. 67A, 267 (70), 70A, 289 (70), T.C. Chen and T. Janssen, Physica 77, 290 (74)
[3] J.L. Richard, Nuovo Cim. 8A, 485 (72)
[4] A. Janner and T. Janssen, Physica 60, 292 (72)
[5] E. Brown, Phys. Rev. 133A, 1037 (64); J. Zak, Phys. Rev. 134A, 1602 and 1607 (64), 136A, 776 (64)
[6] M. Boon, J. Math. Phys. 13, 1268 (72)
[7] N. Giovannini, Proc. 3rd Int. Coll. on Group Theor. Methods in Physics 1974, p. 336
[8] A. Janner and E. Ascher, Physica 54, 77 (71)
[9] P.J. McCarthy, Proc. Roy. Soc. 343A, 489 (75)
[10] S. Helgason, "Differential Geometry and Symmetric Spaces", Acad. Press N.Y. and London 1962
[11] E. Thoma, Math. Ann. 153, 11 (64) and Inv. Math. 6, 190 (68)
[12] J. Dixmier, Ann. Inst. Fourier 7, 315 (57)
[13] A. Janner and T. Janssen, Physica 53, 1 (71)
[14] L. Cattareo and A. Janner, J. Math. Phys. 15, 1155 and 1166 (74)
[15] J. Dixmier, "Les algèbres d'operateurs dans l'espace hilbertien", ed. Gauthier Villars, Paris 1969
[16] A.J. Coleman in "Group theory and its applications I", ed. Loebl 1968, p. 57

[17] G.W. Mackey, "Induced representations of groups and quantum mechanics", Benjamin N.Y. 1968

[18] G.W. Mackey, Appendix to I.E. Segal "Mathematical problems of relativistic physics", Amer. Math. Soc., Providence R.I. 1963

[19] G.W. Mackey, Acta Math. 99, 265 (58)

[20] W. Opechowski and W.G. Tam, Physica 42, 529 (69)

[21] N. Giovannini, Chapters IV and V of the present treatise

CHAPTER II

THE NON-RELATIVISTIC CASE AND THE SCHRÖDINGER EQUATION

Introduction

The natural frame for the description of physical systems interacting with electromagnetic forces is, in terms of group theory, the Poincaré group, since the Maxwell equations are Poincaré covariant (and describe, in the absence of sources, relativistic elementary particles of mass zero and helicity one). Nevertheless it is as is well known often of practical significance to consider such a problem in the so-called non-relativistic approximation. In many cases, this approach has revealed itself in the past to be a very useful and successful one. In this chapter we shall therefore analyze the problem of a charged particle in the presence of an external field and obeying the Schrödinger equation with minimal coupling.

The Schrödinger equation is, as is well known, covariant under the Galilei group (the Galilei group being the contraction, for $c \longrightarrow \infty$, of the Poincaré group), since it characterizes a projective unitary irreducible representation of the Galilei group [1] and describes thus a Galilean (massive) elementary particle (in the sense of

Wigner [2]). Hence it may seem at first sight that the largest kinematical symmetry group which may be taken into consideration when an external field is present is the intersection of the Galilei group (transforming the Schrödinger equation covariantly) with the Poincaré group (acting covariantly on the external field equations). This group consists obviously of Euclidian transformations, time translations and time reversal and will be denoted here as the Shubnikov group [3]. This point of view may be a to restrictive one however: the situation is not that simple, as it is not evident (and not true) that there are no e.m. fields that can be, in a consistent and physically meaningful way, described in a (non-relativistic) Galilean invariant theory. Neglecting this possibility would anyway contain the risk that meaningful symmetries and thus useful physical information could have been ignored.

In the absence of sources, non-relativistic Maxwell equations have been obtained by Lévy-Leblond [1] as describing Galilean elementary particles of mass zero and helicity one. From their structure, it is, in particular possible to see which are the properties of the Maxwell equations and of the electromagnetic fields which are purely relativistic and which ones are not (this analysis may in fact also usefully be extended to the case where sources are present). We shall discuss in more detail in section one the conditions an e.m. field has to satisfy in order to have meaningful exact (or approximate) Galilean symmetries not of the Shubnikov type. When these conditions are not satisfied it is clear that, as explained above, only Shubnikov transformations may be considered. As however the Shubnikov group is a subgroup of the Galilei group, the results concerning it follow trivially from the corresponding more general Galilean case that we shall then consider in more detail.

The purpose of this chapter is essentially the following: we want to investigate if the methods we used in the relativistic case [4] may be extended for the determination of invariance operator groups, based on space-time symmetries of e.m. fields, for the Schrödinger equation and to see if their structure properties undergo or not some essential change. Although this kind of program could in principle be fulfilled by taking properly the non-relativistic limit of the previous results, we thought it preferable, also because this kind of limit procedure is in fact not trivial and often unclear, to reformulate the theory in an exact Galilean frame. On the other side, there have been in the last decade various examples of rather surprising results of Galilean physics as studied with methods parallel to the ones that had shown to be successful in the (Einsteinian) relativistic context, and this is also a reason for the approach followed here.

This Chapter will be organized as follows: in the first part we define the class of fields we consider and analyze their properties, in particular their Galilean symmetry groups and their general relationship to the Galilei group. In the second part we then construct explicitely symmetry groups for a corresponding class of potentials (one potential for each field, chosen in a convenient way). Different choices of potentials give rise to isomorphic symmetry groups. Because of their particular physical interest we consider also in more detail the case of periodic fields. In the third part we then construct, also explicitely, invariance operator groups based on the symmetries of the field and commuting with the Schrödinger operator. The essential properties of these groups are then discussed and related to the ones of the corresponding (Einsteinian) relativistic problem [4].

Since the results are often analogous to the Poincaré case, details have been correspondingly omitted and the accent is essentially put on parts which needed a more precise reformulation.

1. Galilean electromagnetic fields and their symmetry groups.

Let us first define briefly, similarly as in the Poincaré case, what we mean more precisely by a Galilean e.m. field, i.e. what is the minimal set of conditions that we shall assume to be fulfilled.

A Galilean e.m. field is a continuous differentiable cross-section over the Galilean space-time GM(4) of the space $\mathbb{R}^3 \times \mathbb{R}^3$ of real vectors $(\vec{E}, \vec{B})$ satisfying the following three conditions

(i) it obeys the homogeneous non relativistic (n.r.) Maxwell equations, i.e. $\forall\ x \in GM(4)$

$$\begin{aligned} (\vec{\nabla}\wedge\vec{E})(x) + \frac{1}{c}\frac{\partial}{\partial t}\vec{B}(x) &= 0 \\ (\vec{\nabla},\vec{B})(x) &= 0 \end{aligned} \tag{1.1}$$

the second set being considered, as in [4], as a definition for the "4-current" vector $(c\rho,\vec{j})$:

$$\begin{aligned} (\vec{\nabla},\vec{E})(x) &\overset{\mathrm{def}}{=} 4\pi\ \rho(x) \\ (\vec{\nabla}\wedge\vec{B})(x) &\overset{\mathrm{def}}{=} \frac{4\pi}{c}\ \vec{j}(x) \end{aligned} \tag{1.1)'$$

These equations need of course some more comment, we shall return to them in a moment. Let us however note here that the presence of the constant c in these equations could be surprising in a non-relativistic context, but is obviously only due to the choice of (gaussian) electromagnetic units we do. The reasons for such a choice lie in that these units are quite commonly used on one side that it will make comparison with the results of [4] easier. Correspondingly an element $x \in GM(4)$ is given by $(x_o, \vec{x})$ with $x_o = c\,t$, t the time. This constant c will then appear also in the explicit form of the Galilei transformations where it will thus be, as everywhere in this Chapter, nothing else than a convenient constant, having nothing to do with the propagation velocity of e.m. Galilean waves, which is infinite.

(ii) it can be expanded as a Fourier integral over the dual space $GM^*(4)$

$$(\vec{E}, \vec{B})(x) = \int d^4k\ (\hat{\vec{E}}, \hat{\vec{B}})(k) \exp(ikx) \tag{1.2}$$

with $k \in GM^*(4)$, $kx \equiv - k_o x_o + \vec{k}.\vec{x}$, and d^4k is the Lebesgue measure. This means that we assume the inverse Fourier integral to exist (as a generalized function):

$$(\hat{\vec{E}}, \hat{\vec{B}})(k) = (\frac{1}{2\pi})^4 \int d^4k\ (\vec{E}, \vec{B})(x) \exp(-ikx) \qquad (1.2)'$$

(iii) it transforms under an element $g = (t, L(\varepsilon,\vec{v},\alpha))$ of the Galilei group $\mathcal{G}$ according to (see e.g. [3]):

$$(g(\vec{E}, \vec{B}))(x) = (g\vec{E}, g\vec{B})(x)$$

with

$$\begin{aligned}(g\vec{E})(x) &\overset{\text{def}}{=} \alpha\vec{E}(g^{-1}x) - \frac{\vec{v}}{c} \wedge \alpha\vec{B}(g^{-1}x) \\ (g\vec{B})(x) &\overset{\text{def}}{=} \varepsilon\ .\ \det\alpha\ .\ \alpha\vec{B}(g^{-1}x) \qquad ,\end{aligned} \tag{1.3}$$

the transformation $g = (t, L(\varepsilon,\vec{v},\alpha))$ of (1.3), with $t = (t_o,\vec{t}) \in \mathbf{T}$ a 4-translation, $\varepsilon = \pm 1$, $\alpha \in O(3)$, being defined by its action on $x = (x_o,\vec{x}) \in GM(4)$ as

$$
\begin{aligned}
gx_o &= \varepsilon x_o + t_o \\
g\vec{x} &= \frac{\vec{v}}{c} x_o + \alpha\vec{x} + \vec{t}
\end{aligned}
\qquad (1.4)
$$

i.e. in matrix form

$$(gx)_i = (L)_{ij}\, x_j + t_i \qquad , \quad i,j = 0,1,2,3$$

where

$$L = \begin{pmatrix} \varepsilon & 0 \\ \vec{v}/c & \alpha \end{pmatrix}$$

and L is called an homogeneous Galilei transformation.

Together with the transformation laws (1.3) the n.r. Maxwell equations (1.1) can be verified to be exactly covariant under the Galilei group [1, 5, 6]. Let us thus discuss them now in some more detail. When the source terms vanish, these equations describe an irreducible representation of the Galilei group and define thus (group theoretically) Galilean elementary particles of mass zero and helicity one [1], in a perfectly analogous way as the Maxwell equations do in the relativistic case. The only difference with the usual Maxwell equations is the absence in the second part of (1.1)' of the famous displacement current term introduced by Maxwell himself (and which gives their well known relativistic importance to his equations). This implies that it will be consistent to consider Galilean symmetries of the external field only if this displacement current vanishes (as for static electric fields). Furthermore it will make sense to consider Galilean transformations as approximate symmetries when this term is of negligible importance (think of low frequency problems). We note further here that the original symmetry of the Maxwell equations

$$(\vec{E},\vec{B}) \leftarrow -- \rightarrow (\vec{B},-\vec{E})$$

still reflects itself in this non-relativistic frame and that the following alternative n.r. Maxwell equations are also covariant under the Galilei group

$$
\begin{aligned}
(\vec{\nabla}\wedge\vec{E})(x) &= 0 \\
(\vec{\nabla},\vec{B})(x) &= 0 \\
(\vec{\nabla},\vec{E})(x) &= 4\pi\ \rho(x) \\
(\vec{\nabla}\wedge\vec{B})(x) - \frac{1}{c}\frac{\partial}{\partial t}\vec{E}(x) &= \frac{4\pi}{c}\vec{j}(x)
\end{aligned}
\tag{1.5}
$$

with now, in place of (1.3)

$$
\begin{aligned}
(g\vec{E})(x) &= \alpha\vec{E}(g^{-1}x) \\
(g\vec{B})(x) &= \varepsilon\,.\det\alpha\,(\alpha\vec{B}(g^{-1}x) + \frac{\vec{v}}{c}\wedge\alpha\vec{E}(g^{-1}x))
\end{aligned}
\tag{1.6}
$$

The same argumentations as above may now be applied to this case as well, the role of the displacement current being replaced by the Faraday term $\frac{\partial\vec{B}}{\partial t}$ (and electrostatics by magnetostatics).

In the case where the "4-current"$(c\rho,\vec{j})$ defined by (1.1)' does not vanish additional conditions do arise as, from the Galilean invariance of these equations, this "4-current" transforms like a covariant vector and not like a contravariant one as it should physically, this behaviour implying that $\vec{j}$ cannot be identified with moving charges [5,6] but is only to be considered as a source for the magnetic field. This is also illustrated by the fact that the r.h. sides of (1.1)' do not satisfy the usual continuity equation but rather

$$(\vec{\nabla},\vec{j})(x) = 0 \qquad (1.7)$$

and in addition, with $g \in \mathcal{G}$

$$\int d^3x(1-g)\rho(x) = 0 \qquad (1.7)'$$

i.e. only stationary magnetic sources may be considered whereas the total charge is independent of the Galilean frame of reference [5]. In the other case (1.5) the current does transform in a physically meaningful way, the difficulty being then that the magnetic field $\vec{B}$ has no effect on the current distribution of the external field (no Lorentz-like force).

These (expected) restrictions make quite clear that it will strongly depend on the specific physical situation considered whereas or not it is physically meaningful to consider (partly or as a whole) Galilean transformations in defining the symmetry group of the field, and in an exact or in an approximate sense. Nevertheless, as it has also been shown in [5], the class of physical fields that can be consistently described in a pure Galilean framework is quite a large one and this makes at our opinion worthwhile not to restrict ourselves to Shubnikov transformations only, in our general group theoretical approach.

The transformation properties (1.3) define, as in the Poincaré case a symmetry group G_{EB} of the e.m. field which has the following properties

<u>Proposition 1.1.</u> G_{EB} is a closed subgroup of the Galilei group $\mathcal{G}$ and appears as an extension of U_{EB} (its pure translational part) by a subgroup K_{EB} of the homogeneous Galilei group K. The following diagram of groups has exact rows and is commutative (morphism of group extensions)

$$\begin{array}{ccccccccc} 0 & \dashrightarrow & U_{EB} & \longrightarrow & G_{EB} & \dashrightarrow & K_{EB} & \longrightarrow & 1, \quad m,\varphi \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & \mathcal{G} & \longrightarrow & K & \longrightarrow & 1, \quad 0,\varphi \end{array} \tag{1.8}$$

where m is a 2-cocycle ($m \in Z^2_\varphi(K_{EB},U_{EB})$) and φ is defined as the natural homomorphism of K into Aut T (respectively its restriction on K_{EB} into Aut U):

$$(\varphi(L)t)_i \;=\; L_{ij}t_j \qquad i,j = 0,1,2,3 \quad . \tag{1.9}$$

The bottom extension of (1.8) is, as is well known, split (i.e. $\mathcal{G}$ can be written as semidirect product of its translations subgroup T by the homogeneous part K).

Similarly as in [4] we may decompose also G_{EB} in connected and discrete parts in the following way:

<u>Proposition 1.2.</u> Let G be any closed subgroup of the Galilei group, G^c its connected component of the identity, then G appears in the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \dashrightarrow & G^c & \dashrightarrow & G & \dashrightarrow & G^d & \dashrightarrow & 1, \quad n,\ \xi \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \dashrightarrow & G^c & \dashrightarrow & N(G^c) & \dashrightarrow & N(G^c)/G^c & \dashrightarrow & 1, \quad n',\xi \end{array} \tag{1.10}$$

where $N(G^c)$ is the normalizer of G^c in $\mathcal{G}$ and the factor systems n and n', so as the mapping ξ, are defined as in Proposition 1.1 of [4].

Using this decomposition (1.10), the Propositions 1.2 and 1.4 of [4] remain, up to the obvious changes, true. As the proofs of these results and of the two above Propositions are the same as in the Poincaré case, they are omitted here.

Again because of the important role discrete translational symmetries are going to play in the sequel, and because of their particular physical interest, we define further a periodic field by the additional condition

(iv) A Galilean e.m. field is called periodic if and only if the distribution (1.2)' can be written as the sum over a (n_σ-dimensional) lattice $\Lambda^* \subseteq GM^*(4)$ of Dirac δ-functions (with constant uniform factors), i.e. if and only if (1.2) can be written as

$$(\vec{E},\vec{B})(x) = \sum_{K\in\Lambda^*} (\hat{\vec{E}},\hat{\vec{B}})(K) \exp(iKx) \quad . \tag{1.11}$$

In terms of these Fourier coefficients the transformation law (1.3) can be written as

$$(g\vec{E},g\vec{B})(x) = \sum_{K'\in(\Lambda^*)'} (\hat{\vec{E}}',\hat{\vec{B}}')(K') \exp(iK'x) \tag{1.12}$$

where $\hat{\vec{E}}'(K')$ and $\hat{\vec{B}}(K')$ are straightforwardly obtained to be given by, for $i = 1,2,3$

$$\hat{E}'_i(K') = \left[\sum_j \alpha_{ij}\hat{E}_j(\tilde{L}^{-1}K') - \sum_{k,l,m} \varepsilon_{ik\ell} \frac{v_k}{c} \alpha_{\ell m}\hat{B}_m(\tilde{L}^{-1}K')\right] e^{-iK't} \tag{1.13}$$

$$\hat{B}'_i(K') = \varepsilon \det\alpha \cdot \sum_i \alpha_{ij}\hat{B}_j(\tilde{L}^{-1}K')\, e^{-iK't}$$

with $\varepsilon_{ik\ell}$ the completely antisymmetric tensor of order 3, $K' \equiv \tilde{L}K \in \tilde{L}\Lambda^* \equiv (\Lambda^*)'$, the rotated lattice and $\tilde{L}$ is defined by the condition

$$K \cdot L^{-1}x = \tilde{L}K \cdot x \qquad \forall K\in GM^*(4), \forall x\in GM(4) \ . \tag{1.14}$$

It follows from (1.14) that, for $L = L(\varepsilon,\vec{v},\alpha)$ we have, in matrix form

$$\tilde{L} = \begin{pmatrix} \varepsilon & \varepsilon.\alpha^{-1}\vec{v}/c \\ 0 & \alpha \end{pmatrix} \tag{1.15}$$

Note that the mapping $\sim : L \longrightarrow \tilde{L}$ is homomorphic and defines even an isomorphism. This implies that in the dual space (k-space), the mapping

$$\tilde{L} : GM^*(4) - \quad \longrightarrow GM^*(4) \tag{1.16}$$

is a continuous automorphism, so that the formulas (1.13) can be extended to Fourier integrals.

It follows also from (1.13) that an element $g = (t, L(\varepsilon,\vec{v},\alpha)) \in G$ is a symmetry for the field in (1.11) if and only if

$$\text{(i)} \quad \tilde{L}.\Lambda^* = \Lambda^*$$

$$\text{(ii)} \quad \hat{\vec{E}}(K) = \left[\alpha\hat{\vec{E}}(\tilde{L}^{-1}K) - \left(\frac{\vec{v}}{c}\wedge\alpha\hat{\vec{B}}(\tilde{L}^{-1}K)\right)\right] e^{-iKu(L)}, \forall K \in \Lambda^* \tag{1.17}$$

$$\text{(iii)} \quad \hat{\vec{B}}(K) = \varepsilon.\det\alpha.\ \hat{\vec{B}}(\tilde{L}^{-1}K)\ e^{-iKu(L)}, \forall K \in \Lambda^*$$

where $t \equiv a + u(I)$, a a translational symmetry (satisfying thus $Ka = 2n\pi$, $n \in \mathbb{Z}$, $\forall K \in \Lambda^*$) and $u(L)$ is a translation associated to L (and called, as in the Poincaré case [7], non-primitive). The above formulas may appear tedious but will be very useful in the sequel for the determination of the symmetry groups of the potentials we are going to define now.

2. Potential symmetry groups.

As in the relativistic case, Galilean transformations which leave a field invariant are not, in general, symmetries for a corresponding potential and since the Schrödinger equation depends on the e.m. field only via a potential, it will again be necessary to combine gauge and space-time transformations in order to retain the physical information given by the symmetries of the external field.

Let thus $(\vec{E},\vec{B})(x)$ be some (arbitrary but fixed) e.m. field as in section 1 (but not necessarily periodic). Since it satisfies the n.r. Maxwell equations (1.1), there exists a 4-potential $A(x) \equiv (\phi(x),\vec{A}(x))$ such that

$$\begin{aligned} \vec{E}(x) &= -\vec{\nabla}\phi(x) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(x) \\ \vec{B}(x) &= \vec{\nabla}\wedge\vec{A}(x) \quad . \end{aligned} \tag{2.1}$$

As is well known, the equations (2.1) determine $A(x)$ up to an arbitrary gauge transformation $\chi(x)$ so that the field is equivalently described by the two potentials A and A' related by the following transformation

$$\begin{aligned} \phi'(x) &= \phi(x) - \frac{1}{c}\frac{\partial}{\partial t}\chi(x) \\ \vec{A}'(x) &= \vec{A}(x) + \vec{\nabla}\chi(x) \end{aligned} \tag{2.2}$$

Because of this equivalence, there is a certain freedom in the choice of the transformation law of a potential under the Galilei group, the only requirement being that under any element $g \in \mathcal{G}$ the mapping

$$g : A \longrightarrow gA \tag{2.3}$$

is such that the following diagram is commutative

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[d, "(2.1)"'] & & gA \arrow[d, "(2.1)"] \\ (\vec{E},\vec{B}) \arrow[rr, "(1.3)"] & & (g\vec{E},g\vec{B}) \end{tikzcd} \tag{2.4}$$

Using elementary formulas of vector calculus, it is straightforward to verify that the following choice satisfies the requirement (2.4):

$$(gA)(x) = (g\phi(x),\ g\vec{A}(x))$$

with

$$\begin{aligned} (g\phi)(x) &\stackrel{\text{def}}{=} \phi(g^{-1}x) + \frac{\vec{v}}{c}\,\alpha\,\vec{A}(g^{-1}x) \\ (g\vec{A})(x) &\stackrel{\text{def}}{=} \varepsilon\,.\,\alpha\,\vec{A}(g^{-1}x) \end{aligned} \tag{2.5}$$

The most general choice is then an arbitrary combination of the transformations in (2.5) and in (2.2). Similarly as in the Poincaré case [3,4] we may now choose, for each $g \in G_{EB}$, and for each (arbitrary but fixed) potential A satisfying (2.1), a gauge function $\chi_g(x)$ in such a way that the combined transformation leaves A invariant. This gauge function is then called compensating (for A). The group Q generated by these pairs is called the symmetry group of the potential and has exactly as in the relativistic case, the following structure:

<u>Proposition 2.1.</u> The symmetry group Q of the potential A appears in the following commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r] \arrow[d, hook] & Q \arrow[r] \arrow[d, "\mu"] & G_{EB} \arrow[r] \arrow[d, hook] & 1, & f,\psi \\ 0 \arrow[r] & J \arrow[r] & J\mathcal{G} \arrow[r] & \mathcal{G} \arrow[r] & 1, & 0,\psi \end{tikzcd} \tag{2.6}$$

where $\mathcal{F}$ is the subgroup of $\mathcal{R}$ generated by all $f(g_i,g_j)$, $\forall g_i,g_j \in G_{EB}$ defined by

$$f(g_i,g_j) = \chi_{g_i}(x) + \psi(g_i)\chi_{g_j}(x) - \chi_{g_i g_j}(x) \quad \in \quad \mathcal{R} \, . \qquad (2.7)$$

Further J is the abelian group of the (allowed) gauge functions and $\psi(g) \in \text{Aut } J$ is given, $\forall g = (t,L(\varepsilon,\vec{v},\alpha)) \in \mathcal{G}$ and $\forall \varphi(x) \in J$

$$\psi(g)\varphi(x) = \varepsilon\,\varphi(g^{-1}x) \qquad . \qquad (2.8)$$

The gauge functions $\chi_g(x)$ in (2.7) are defined, up to a constant, by

$$\begin{aligned} - \partial_o \chi_g(x) &= (g\phi)(x) - \phi(x) \\ \vec{\nabla} \chi_g(x) &= (g\vec{A})(x) - \vec{A}(x) \end{aligned} \qquad (2.9)$$

Let us also remind here that the symmetry groups of different potentials related by a gauge transformations give rise to equivalent extensions (2.6) and hence are isomorphic [3].

For the other symbols and properties of Q we refer to [3,4,8].

Our purpose is now to construct this group explicitely for an arbitrary field, choosing in some convenient way a corresponding potential. In order to illustrate the method and to make the results more explicit, we consider first periodic fields only.

a) periodic fields

Let thus $(\vec{E},\vec{B})(x)$ be some periodic field as in (1.11). We first want to examine the conditions under which the potential $A = (\phi,\vec{A})$ can be chosen periodic too, i.e. can be expanded as a Fourier series over the same lattice Λ^* as the field

$$(\phi,\vec{A})(x) = \sum_{K\in\Lambda^*}(\hat{\phi},\hat{\vec{A}})(K)\exp(iKx) \quad . \tag{2.10}$$

Using (2.1) this implies

$$(\hat{\vec{E}},\hat{\vec{B}})(K) = (-i\vec{K}\hat{\phi}(K) + \frac{i}{c}K_o\hat{\vec{A}}(K), i\vec{K}\wedge\hat{\vec{A}}(K)) \tag{2.11}$$

with $K = (K_o,\vec{K})$, $\vec{K}$ the (dual) space component of K. It follows from (2.11) that necessarily $\hat{\vec{B}}(\vec{K}=0) = \hat{\vec{E}}(K=0) = 0$. This condition can be seen to be also sufficient: indeed let first $\vec{K} \neq 0$, then we find, after scalar, respectively vector multiplication from the left of (2.11):

$$\vec{K}\wedge\hat{\vec{B}}(K) = i\vec{K}\wedge(\vec{K}\wedge\hat{\vec{A}}(K)) = i\vec{K}(\vec{K},\hat{\vec{A}}(K)) - i\hat{\vec{A}}(K)\vec{K}^2$$

and

$$(\vec{K},\hat{\vec{E}}(K)) = -i\vec{K}^2\hat{\phi}(K) + \frac{i}{c}K_o(\vec{K},\hat{\vec{A}}(K)) \quad .$$

So that, for given $\hat{\vec{E}}(K)$, $\hat{\vec{B}}(K)$ we may define, $\forall K \in \Lambda^*|\vec{K} \neq 0$

$$\begin{aligned}\hat{\vec{A}}(K) &= i\,\frac{\vec{K}\wedge\hat{\vec{B}}(K)}{\vec{K}^2} + \vec{K}\,\frac{(\vec{K},\hat{\vec{A}}(K))}{\vec{K}^2} \\ \hat{\phi}(K) &= i\,\frac{(\vec{K},\hat{\vec{E}}(K))}{\vec{K}^2} + \frac{K_o}{c}\,\frac{(\vec{K},\hat{\vec{A}}(K))}{\vec{K}^2}\end{aligned} \tag{2.12}$$

As is easily seen the second part of these expressions can be absorbed by a gauge transformation so that we may restrict ourselves to the first part only.

Let further $\vec{K} = 0$, $K_o \neq 0$ then necessarily $\hat{\vec{B}}(K) = 0$ as follows from the Maxwell equations (1.1) for the corresponding Fourier coefficients. We may thus directly define $\forall K \in \Lambda^*|\vec{K} = 0 \;\; K_o \neq 0$

$$\hat{\vec{A}}(K) = \frac{c}{iK_o}\hat{\vec{E}}(K)$$
$$\hat{\phi}(K) = 0 \qquad (2.13)$$

and the potential so constructed is periodic and satisfies conversely (2.1). Using these results we may now formulate the following:

<u>Proposition 2.2.</u> Given any periodic e.m. field as in section 1 (1.11), there exists a periodic potential A if and only if the Fourier coefficients of the field at the origin of $GM^*(4)$ vanish.

<u>b) arbitrary fields</u>

Let us generalize now the previous treatment for an arbitrary Galilean e.m. field satisfying the conditions of section 1. Let therefore $(\vec{E},\vec{B})(x)$ be such a field: we want to construct a standard convenient map π, as in [4], with

$$\pi : (\vec{E},\vec{B})(x) \longrightarrow (\phi,\vec{A})(x) \qquad (2.14)$$

i.e. we want to define a convenient representative in the set of all potentials related to this given field. Let therefore S denote again the spectrum of the field, i.e. the set of 4-vectors $k \in GM^*(4)$ for which (1.2)' is not zero. We consider then the following independent parts of the field:

(o) Let $k \in S$ and $k = 0$, we define

$$(\vec{E}^{(o)}, \vec{B}^{(o)})(x) = \int d^4k\ (\hat{\vec{E}}^{(o)}(k), \hat{\vec{B}}^{(o)}(k))\ \exp\ (ikx) \qquad (2.15)$$

with

$$(\hat{\vec{E}}^{(o)}, \hat{\vec{B}}^{(o)})(k) = \begin{cases} (\hat{\vec{E}}, \hat{\vec{B}})(k) & \text{for } k = 0 \\ 0 & \text{else} \end{cases} \tag{2.16}$$

This field can be expressed as a finite polynomial in the components of x (as distributions concentrated at a point are necessarily finite linear combinations of δ-functions and derivatives [9]). We assume here however for simplicity and because of its physical interest that the field in (2.15) is constant uniform.

(1) Let $k \in S$ and $\vec{k} = 0$, $k_o \neq 0$. It is no longer necessarily true that $\hat{\vec{B}}(k) = 0$ for this part of the spectrum, as it was for periodic fields. Let us therefore define in a first step the following

$$(\vec{E}^{(1)}, \vec{B}^{(1)})(x) = \int d^4k \; (\hat{\vec{E}}^{(1)}, \hat{\vec{B}}^{(1)})(k) \exp ikx \tag{2.17}$$

where $(\hat{\vec{E}}^{(1)}, \hat{\vec{B}}^{(1)})(k)$ are defined analogously as in (2.16). The magnetic part of this field induces because of Maxwell equation an own electric field. Assuming again for simplicity that $\vec{B}^{(1)}(x)$ depends only on time (the lowest order term) this electric field can be written, using (1.1), as

$$\vec{E}_a^{(1)}(x) = -\frac{1}{2c} \frac{\partial}{\partial t} (B^{(1)}(t) \wedge \vec{x}) \quad . \tag{2.18}$$

The rest of the electric field corresponding to this part of the spectrum may then be written

$$\vec{E}_b^{(1)}(x) = \int d^4k \; (\hat{\vec{E}}^{(1)}(k) - \hat{\vec{E}}_a^{(1)}(k)) \exp ikx \tag{2.19}$$

(2) Let finally $S^{(2)} = \{k \in S \,|\, \vec{k} \neq 0\}$. For this part of the spectrum the generalization of (2.12) is straightforward and we may use the same gauge as before, so that all together we now can define (2.14) explicitly by

$$\pi(\vec{E},\vec{B})(x) \stackrel{\text{def}}{=} A^{(o)}(x) + A^{(1)}(x) + A^{(2)}(x)$$

with

$$\phi^{(o)}(x) = - \tfrac{1}{2} \, (\vec{E}^{(o)}, \vec{x})$$

$$\phi^{(1)}(x) = 0 \tag{2.20}$$

$$\phi^{(2)}(x) = \int d^4k \; i \, \frac{(\vec{k}, \hat{\vec{E}}^{(2)}(k))}{\vec{k}^2} \exp (ikx)$$

and respectively

$$\vec{A}^{(o)}(x) = \tfrac{1}{2} \, (\vec{B}^{(o)} \wedge \vec{x}) - \tfrac{1}{2} \, \vec{E}^{(o)} x_o$$

$$\vec{A}^{(1)}(x) = \tfrac{1}{2} \, (\vec{B}^{(1)}(t) \wedge \vec{x}) + \int d^4k \, \frac{c}{ik_o} \, \hat{\vec{E}}_b^{(1)}(k) \exp (ikx) \tag{2.21}$$

$$\vec{A}^{(2)}(x) = \int d^4k \; i \left(\frac{\vec{k} \wedge \hat{\vec{B}}^{(2)}(k)}{\vec{k}^2} \right) \exp (ikx)$$

Note that the above decomposition of the spectrum of the field in disconnected parts is invariant under the Galilei group, as follows from the action defined in (1.15). Using now Maxwell equations and elementary vector calculus, it is again straightforward to verify that the potentials defined in (2.20) and (2.21) satisfy (2.1). Since this calculation does not present any difficulty we drop the details.

The purpose of this ansatz is of course that it makes it now possible to find explicitely, $\forall \, g \in G_{EB}$, the corresponding compensating gauges as defined, up to a constant, by (2.9), and the important factor system (2.7) which determines, from G_{EB}, the structure of the symmetry group of the potential. After a tedious but straightforward calculation, using in particular the invariance property of the field and the Galilei invariance of the Lebesgue measure d^4k (with respect to the action (1.15)) we find for the various parts of the field separately, and for $g = (t, L(\varepsilon, \vec{v}, \alpha)) \in G_{EB}$, $t = (t^o, \vec{t})$

$$
\begin{aligned}
\chi_g^{(o)}(x) &= -\tfrac{1}{2}\,(\vec{E}^{(o)}, \vec{t})x_o - \tfrac{1}{2}\Big((\vec{B}^{(o)} \wedge \vec{t}) - \vec{E}^{(o)}t^o, \vec{x}\Big) + c_o \\
\chi_g^{(1)}(x) &= c_1 \qquad\qquad (2.22) \\
\chi_g^{(2)}(x) &= c_2
\end{aligned}
$$

where c_o, c_1, c_2 are free constants which may be chosen zero so to satisfy the condition that the map μ in (2.6) is a monomorphism. The factor system f of the top extension in (2.6) is then obtained from (2.7) and (2.22), again for the various parts of the field separately, as equal to

$$
\begin{aligned}
f^{(o)}(g_1, g_2) &= \tfrac{1}{2}\,(\vec{E}^{(o)}, \vec{t}_2)t_1^o + \tfrac{1}{2}\left(\vec{B}^{(o)} \wedge \alpha_1\vec{t}_2 - \alpha_1\vec{E}^{(o)}t_2^o, \vec{t}_1 - \frac{\vec{v}_1}{c}\,t_1^o\right) \\
f^{(1)}(g_1, g_2) &= f^{(2)}(g_1, g_2) = 0 \qquad\qquad (2.23)
\end{aligned}
$$

where $g_i = (t_i, L(\varepsilon_i, \vec{v}_i, \alpha_i))$, $t_i = (t_i^o, \vec{t}_i)$, $i = 1,2$.

Since Galilei transformations acting on potentials and our map π acting on fields are both linear, the compensating gauges and the factor system for the whole of the field are again obtained by simple addition of the various parts so that finally

$$\chi_g(x) = \chi_g^{(o)}(x)$$
$$f(g_1,g_2) = f^{(o)}(g_1,g_2) \qquad (2.24)$$

The first result will characterize the imbedding of the symmetry group of the potential in the invariance operator group to be defined in the next section, whereas the second characterizes the structure of this symmetry group itself.

Compared with the results we had found in [4] (relativistic problem) we note that the only essential modification is that there is no field anymore (in the class we consider) for which the compensating gauges are inequivalent to zero whereas the corresponding factor system vanishes (as it was the case for radiation fields previously).

Let us now consider the restriction of the factor system (2.24) on $U_{FB} \times U_{EB}$, where U_{EB} is the translation subgroup of G_{EB}. One obtains easily from (2.24), with $t_1,t_2 \in U_{EB}$

$$f(t_1,t_2) = \tfrac{1}{2}(\vec{E}^{(o)},\vec{t}_2)t_1^o - \tfrac{1}{2}(\vec{E}^{(o)},\vec{t}_1)t_2^o - \tfrac{1}{2}(\vec{B}^{(o)},\vec{t}_1\wedge\vec{t}_2) \qquad (2.25)$$

which has the following useful properties

$$\text{(i)} \quad f(t_1,t_2) = -f(t_2,t_1)$$
$$\text{(ii)} \quad f(t_1,\lambda t_2) = f(\lambda t_1,t_2) = \lambda f(t_1,t_2) \;, \forall\lambda \in \mathbb{R} \qquad (2.26)$$
$$\text{(iii)} \quad f(t_1,t_2+t_3) = f(t_1,t_2) + f(t_1,t_3)$$

From the first of these properties it follows that f is equivalent to zero on $U_{EB} \times U_{EB}$ (as a factor system) if and only if it identically vanishes, since a trivial factor system is necessarily symmetric,

U_{EB} being abelian. Together with (2.24) this implies that whenever the set of all non-primitive translations associated to K_{EB} can be chosen in the real space spanned by U_{EB} (as is always possible for a periodic field for example), f is equivalent to zero on $G_{EB} \times G_{EB}$ if and only if it vanishes identically on $U_{EB} \times U_{EB}$. The two other properties in (2.26) imply that f is then completely known when it is known on a space-time basis $\{b\}$ for U_{EB}. These properties will be helpful in the sequel.

Having derived the explicit form of the symmetry groups of the potentials we can now turn to the problem of the invariance operator groups of the Schrödinger equation. It is however perhaps worthwhile noting here that again it turns out that these symmetry groups of the potentials are isomorphic to the corresponding symmetry groups of the fields whenever the contribution of the Fourier transform at the origin of the dual space vanishes, i.e. for the fields we consider, whenever the field has no constant uniform part.

3. Invariance operator groups of the Schrödinger equation.

Let us now consider a charged (spinless) massive particle moving in some arbitrary (but fixed) Galilean e.m. field and obeying the Schrödinger equation with minimal coupling. Our purpose is now to define similarly as in the relativistic case invariance operator groups commuting with the Schrödinger operator and based on the symmetries of the field.

Let therefore $\mathcal{H}$ denote the (separable) carrier Hilbert space of physical states, and $q \in Q$, an element of the symmetry group of the corresponding potential as defined in the previous section. Using the exactness of (2.6) we may write q as a pair (f,g) with $f \in \mathcal{F}$ and $g = (t, L(\varepsilon, \vec{v}, \alpha)) \in G_{EB}$. We now define the imbedding of Q in the group $U(\mathcal{H})$ of unitary/antiunitary operators on $\mathcal{H}$ by the following mapping ϕ_b

$$\phi_b : Q \longrightarrow U(\mathcal{H})$$

with, for $(f,g) \in Q$,

$$\phi_b(f,g) \overset{\text{def}}{=} \exp\left\{-i \frac{e}{c\hbar} (f + \chi_g(x) + \psi_g(x))\right\} P_g K_\varepsilon \overset{\text{def}}{=} \langle f,g\rangle_b \ .(3.1)$$

In (3.1), P_g is the substitution operator, e the charge of the particle. $\hbar$ the Planck's constant divided by 2π and K_ε is given by

$$K_\varepsilon = \begin{cases} K & , \text{ if } \varepsilon = -1 \\ 1 & , \text{ if } \varepsilon = 1 \end{cases} \tag{3.2}$$

with K the complex conjugation operator. The gauge functions $\chi_g(x)$ and $\psi_g(x)$ in (3.1) are respectively given by (2.24) and by

$$\frac{e}{c\hbar} \psi_g(x) \overset{\text{def}}{=} \frac{1}{2c} m \vec{v}^2 x_o - \varepsilon \cdot m(\vec{v}, \vec{x})$$

the latter being the well known [1,3,10] gauge transformation belonging to the Schrödinger equation and expressing the fact that this equation characterizes a (true) projective representation of the Galilei group. To this gauge transformation corresponds the usual factor system

$$\omega_b(g_1,g_2) \equiv \exp\left\{- i \frac{e}{ch} b(g_1,g_2)\right\} = \tag{3.3}$$

$$= \exp\left\{- im\left[\frac{\vec{v}_1^2}{2} - \frac{1}{2c} (\vec{v}_1 + \varepsilon_2\alpha_1\vec{v}_2)^2 t_1^o + \varepsilon_1\varepsilon_1(\alpha_1\vec{v}_2, \vec{t}_1)\right]\right\}$$

The group product in $\mathcal{Q}_b = \phi_b(Q)$ is then easily obtained from (2.6), (3.1) and (3.3), with $\langle f_i, g_i\rangle_b \in \mathcal{Q}_b$, i = 1,2, as given by

$$\langle f_1,g_1\rangle_b \langle f_2,g_2\rangle_b = \langle f_1+\varepsilon_1 f_2+f(g_1,g_2)+b(g_1,g_2),g_1g_2\rangle_b \quad . \tag{3.4}$$

Let now $H_t \equiv H - i\hbar \frac{\partial}{\partial t}$ be the Schrodinger operator, with

$$H = \frac{\hbar^2}{2m} \left(- i\vec{\nabla} - \frac{e}{c\hbar} \vec{A}(x)\right)^2 + e\phi(x) \tag{3.5}$$

and $A = (\phi,\vec{A}) = \pi(\vec{E},\vec{B})$ given by (2.20) and (2.21). It follows, by construction, that for all $\langle f,g\rangle_b \in \mathcal{L}_b$, we have the important property

$$\langle f,g\rangle_b \, H_t = H_t \, \langle f,g\rangle_b \quad . \tag{3.6}$$

This operator group $\mathcal{L}_b$ is therefore called the <u>invariance operator group</u> of the Schrodinger equation. Its structure is needless to say of essential importance as it is from this group and its representations that it will be possible to extract physical information (about solutions of the equation of motion, selection rules, conserved quantities and so on).

As in the relativistic case, it is then possible to use the decomposition of G_{EB} in connected and discrete parts as defined in (1.10) and consider first separately the operator groups corresponding to the one or the other part respectively. The situation is then very analogous to the one of the Poincaré case, the only differences being due to the additional factor system (3.3). This factor system depends however only on the homogeneous part K of the Galilei group and its role with respect to the structure of $\mathcal{L}_b$ is then very different than the one of the factor system f given in (2.24) and which is due to the presence of an external field. In fact, it is easy to see that replacing $\mathcal{F}$ by $\mathcal{F} + \mathcal{B}$, where $\mathcal{B}$ is the subgroup of $\mathcal{R}$ generated by the factor system b in (3.3) the results of sections 3 to 5 of [4] remain true.

In particular, the study of the problem of a Bloch electron in an external c.u. field (section 5 of [4]) does not undergo any change at all, since the (translational) symmetry group is both a subgroup of the Poincaré and of the Galilei group, and since the factor systems then coincide as is readily verified. For these reasons we do not go here into more details. The general results concerning this Galilean case are then the ones exposed in section 6 of the same paper.

Although a quite detailed reformulation was necessary for the explicit construction of the invariance operator groups of the Schrödinger equation the results concerning these groups are in fact very similar to the ones of the relativistic problem. In particular, we have shown that the factor systems determining the structure of these groups depend again only on the constant uniform part of the field and, even if simple, this result is quite remarkable, especially seen the large class of fields we have considered. Furthermore, also in this case, there is a large class of fields giving rise to pathological non-Type I groups [4]. Because of these closed relationships between the Galilean and the Poincaré cases, we refer to [4] for a more detailed discussion on all these aspects.

References

[1] J.M. Lévy-Leblond, J. Math. Phys. 4, 776 (1963); Thèse de Doctorat Orsay (1965); Comm. Math. Phys. 6, 286 (1967); "Galilei Group and Galilean Invariance "in" Group Theory and Its Applications", Vol. 2 ed. Loebl, Acad. Press NY and London (1971)

[2] E.P. Wigner, Ann. Math. 40, 149 (1939)

[3] A. Janner and T. Janssen, Physica 53, 1 (1971)

[4] N. Giovannini, Chapter I of the present treatise

[5] M. Le Bellac and J.M. Lévy-Leblond, Nuovo Cim. 14B, 217 (1973)

[6] H. Bacry and J. Kubar- André, Int. Journ. Theor. Phys. 7, 39 (1973)

[7] A. Janner and E. Ascher, Physica 54, 77 (1971)

[8] U. Cattaneo and A. Janner, J. Math. Phys. 15, 1155 and 1166 (1974)

[9] I.M. Gel'fand and G.E. Shilov, "Generalized Functions", Vol. 1, Acad. Press NY and London (1964)

[10] V. Bargmann, Math. Ann. 59, 1 (1954)

CHAPTER III

SOLUTIONS OF THE EQUATIONS OF MOTION

Introduction

As an important advantage of the explicit knowledge of an invariance operator group for a linear equation of motion is the fact that, since by definition this operator group commutes with the operator of that equation, the corresponding solutions span a representation space for this group. Conversely, an explicit knowledge of these representations can be used for a convenient, symmetry adapted, ansatz for obtaining solutions, and the problem of the characterization of these solutions and of those properties can in this way be facilitated and sometimes even completely solved.

In the absence of an external e.m. field, or if the invariance operator group is homomorphic to a space-time symmetry group, these representations may be obtained by induction, in the sense of Mackey, from the representations of the invariant translation subgroup.

In other words we can then in general simplify the problem by chosing as ansatz basis functions of the representations of this subgroup. Let us briefly, for illustration, and from this point of view, consider the well known problem of a (non-relativistic) Bloch electron: the translation subgroup is then continuous along the time axis and generates a lattice in space. From the former, the time-dependent Schrödinger equation reduces to an eigenvalue problem for the energy (the character label of the time translation normal subgroup). Note that the irreducible representations of the whole invariance operator group are then also characterized by this energy value, as orthogonal transformations commute with time translations. From the latter, one may choose the solutions in a Bloch form (the Brillouin zone being in this context nothing else than the character space of this discrete translation group, and the Bloch functions the corresponding basis functions), and in this way one can reduce the (differential) Schrodinger equation to a linear algebraic one, in general easier to handle, as is well known. The same procedure may in fact also be used for time dependent problems and for relativistic particles and leads then to concepts such as mass bands [1]. In the more general case, as follows for instance from the first part of [2], one may then choose as ansatz for the solutions a superposition of plane waves and of functions of the (generalized) Bloch type.

If now an external field as in Chapter I or II is present such that the operators corresponding to space-time translations do no longer form an abelian normal subgroup (and this is the case as we have shown previously only if the field carriers a constant uniform (c.u.) part), this procedure does of course no longer apply and related concepts such as dispersion relations or mass bands for example, or the relationship with physical quantities such as momenta (or pseudo-momenta) may loose their significance. This is a phenomenon which is well known in the particular case where a c.u. magnetic field is applied to a Bloch electron: the concept of Brillouin zones disappears and has to be replaced by the one of the so-called magnetic (reciprocal) unit cells. Still however it remains possible, since in all cases the operator subgroups generated by the translational symmetries of the external field are normal subgroups of the corresponding invariance operator groups, to choose as basis functions for the irreducible representations (and correspondingly as ansatz for the solutions) the basis functions of the carrier spaces of the irreducible representations of these subgroups. The general situation is then as follows: corresponding to continuous or discrete translational symmetries, with or without a field phase factor respectively, there are so to say four kinds of representations building up to the irreducible representations of the general case. The relationships between these various "building blocks" and the whole representations follow then from Proposition 4.1 of [2] (and have been illustrated in the same reference on the example of a Bloch electron in a c.u., e.m. field).

It is the purpose of this Chapter to analyze the situation in the non-trivial case where a c.u. field is also present and to derive the general structure of these representations. We consider therefore in turn the continuous and discrete cases separately. In the discrete case we shall for simplicity restrict ourselves to rational fields only (see [2]) even though the method we propose can in fact be extended to the pathological irrational cases, too. We shall describe a convenient and straightforward method for the explicit calculation of the corresponding representations in the presence of a (further arbitrary) external field and for arbitrary translational symmetry groups in space and time. As a result we show what are then the general form, properties and consequences of these representations.

The notation used throughout this chapter is for convenience the one of a relativistic spinless particle. As space time translations form both a subgroup of the Poincaré and of the Galilei group and as they generate no spin factor system, the results remain of course valid in the other cases as well.

1. Continuous translations.

Let us thus first consider the operator group generated by continuous translations only. It follows from our previous results that this operator group $\mathcal{L}$ appears as an extension of the complex unit circle by the m_o-dimensional real vector space, i.e. the following sequence of groups is exact

$$0 \longrightarrow U(1) \longrightarrow \mathcal{L} \longrightarrow \mathbb{R}^{m_o} \longrightarrow 1, \quad \omega_f, \phi \tag{1.1}$$

and is characterized by a factor system ω_f obtained in [2] as

$$\omega_f(t_1,t_2) = \exp\left\{-\frac{ie}{c\hbar} f(t_1,t_2)\right\} = \exp\left\{-\frac{ie}{2c\hbar} F^{(o)}_{\mu\nu}(t_1)^\nu (t_2)^\mu\right\} \in U(1) \tag{1.2}$$

for t_1, $t_2 \in U^c_F \simeq \mathbb{R}^{m_o}$ the continuous pure (primitive) translational symmetries and for $F^{(o)}$ the c.u. part of the (given) external field. The action ϕ of $\mathbb{R}^{m_o}$ on Aut U(1) is thereby easily calculated from the operator group structure (see [2]) and is given by

$$\phi(t)\ \exp\left(-\frac{ie}{c\hbar} f\right) \equiv \langle o,t\rangle\langle f,o\rangle\langle o,t\rangle^{-1} = \langle f,o\rangle \ , \forall f \in \mathbb{R} \tag{1.3}$$

hence ϕ is trivial.

Since $\mathcal{L}$ in (1.1) is a connected Lie group, it is natural and useful to write its Lie algebra: let therefore Π_α, Π_β, ... be the infinitesimal generators of a minimal set e_α, e_β,... generating U^c_F (but not necessarily orthonormal), and I denote the generator of the phase transformation, we have, from the structure of the group

$$\begin{aligned}[\Pi_\alpha,\Pi_\beta] &= [\sum_\mu (e_\alpha)^\mu \Pi_\mu, \sum_\nu (e_\beta)^\nu \Pi_\nu] \\ &= \sum_{\mu,\nu} (e_\alpha)^\mu (e_\beta)^\nu [\Pi_\mu,\Pi_\nu] =\end{aligned} \tag{1.4 (i)}$$

$$= \sum_{\mu,\nu} (e_\alpha)^\mu (e_\beta)^\nu \left(\frac{e}{c\hbar}\right) F^{(o)}_{\mu\nu} \cdot I \qquad (1.4)\ (i)$$

$$\stackrel{\text{def}}{=} \lambda_{\alpha\beta} \cdot I$$

where the indices μ,ν refer to a (fixed) orthonormal basis e_μ, e_ν of the space-time. Further, using [1.3] we have obviously

$$[\Pi_\alpha, I] = 0 \qquad . \qquad (1.4)\ (ii)$$

It follows from these commutation relations that $\mathcal{Q}$ is a m_o+1-dimensional nilpotent Lie group, whose Lie algebra we shall denote here by $\underline{n}$. Its irreducible representations may then be computed using the general theory of Kirillov [3]. Since this theory is exposed in more details in [4] we just remember here the essential steps of the construction of these representations. Let therefore $\underline{n}'$ denote the dual space of $\underline{n}$ and $\nu \in \underline{n}'$ a linear form. The orbit $O_\nu \subsetneq \underline{n}'$ of ν is defined by the action of the coadjoint representation of $\overline{\mathcal{Q}}$ on $\underline{n}'$, ($\overline{\mathcal{Q}}$ denoting the covering group of $\mathcal{Q}$). We then choose in each orbit one element ν and consider a subalgebra $\underline{\ell} \subsetneq \underline{n}$ of maximal dimension such that

$$[\,\underline{\ell}\,,\,\underline{\ell}\,] \subsetneq \text{Ker}\ \nu \qquad (1.5)$$

Such a subalgebra is called subordinate to ν and of maximal dimension or, simply, a <u>real polarisation</u> at ν. Let then $T_{\nu,\underline{\ell}}$ denote the one-dimensional unitary representation of $L = \exp \underline{\ell}$ that is given by

$$T_{\nu,\underline{\ell}}(\exp x) \stackrel{\text{def}}{=} \exp i\,\nu(x) \quad , \qquad x \in \underline{\ell} \qquad (1.6)$$

and $V_{\nu,\underline{\ell}}$ the representation of $\overline{\mathcal{Q}}$ induced from $T_{\nu,\underline{\ell}}$ and defined thus on the complex valued functions $\varphi(L\lambda)$ on the (right) coset space $\overline{\mathcal{Q}}/L$

and which are measurable and squared integrable with respect to the unique (as a class), invariant (under the action of $\bar{\mathcal{Q}}$) measure μ. We have thus

$$V_{\nu,\underline{\ell}}(\bar{q})\,\varphi(L\lambda) = T_{\nu,\underline{\ell}}(\lambda\,.\bar{q}.(\lambda')^{-1})\,\varphi(L\lambda') \tag{1.7}$$

with $\bar{q} \in \bar{\mathcal{Q}}$, $\lambda.\lambda'$ coset representatives and λ' defined by the condition $\lambda\bar{q}(\lambda')^{-1} \in L$.

It follows from [3] that if one takes exactly one element each orbit and follows the just described procedure, one obtains each irreducible unitary representation of $\bar{\mathcal{Q}}$ exactly once. Since $\mathcal{Q}$ is not simply connected, we obtain all its irreducible unitary representations as above by restricting $\underline{n}'$ to the closed subset of linear forms taking on the kernel of the exponential map integer values times 2π [3,5].

Let us now discuss in more detail these representations for our case: a linear form $\nu \in \underline{r}'$ is given by

$$\nu(\frac{e}{ch}\,f\,I+\sum_{\alpha}t^{\alpha}\Pi_{\alpha}) = (\rho,\{k\})\;(\frac{e}{ch}\,f\,I+\sum_{\alpha}t^{\alpha}\Gamma_{\alpha}) \overset{\text{def}}{=} (\frac{e}{ch})\,\rho\,f + \sum_{\alpha}k_{\alpha}t^{\alpha} \tag{1.8}$$

where $\rho \in \mathbb{R}$ and k is some representant of a $((4-m_o)$-dimensional) class $\{k\}$ of 4-vectors k (in the reciprocal space-time $M^*(4)$ resp. $GM^*(4)$), this class being defined as an element of the coset space of the reciprocal space time by Λ^*, where Λ^* is the set of vectors k satisfying the condition

$$k\,.\,t = 0\,, \qquad \forall\, t \in U_F^c \tag{1.9}$$

Since $\mathcal{Q}$ is not simply connected we have in addition the condition

$$\nu\,(\frac{e}{ch}\,I) \in \mathbb{Z}$$

hence $\rho = r(\frac{ch}{e})$, $r \in \mathbb{Z}$ and takes only discrete values.

The action of $\mathcal{L}$ on these linear forms, via the coadjoint representation, can then be obtained, using (1.4) and (1.8), and for an arbitrary element $\langle f,t\rangle \in \mathcal{L}$, as given by

$$\mathrm{coAd}(\langle f,t\rangle)\ (\rho,\{k\}) = (\rho,\{k + t\cdot F^{(o)}\}) \qquad (1.10)$$

where, in o.n. coordinates, $(t\cdot F^{(o)})_\nu \equiv t^\mu F^{(o)}_{\mu\nu}$.

Let then d_ν denote the dimension of a real polarisation at ν. This dimension may be obtained from the general result of Kirillov:

$$d_\nu = \dim\ (\underline{n}) - \tfrac{1}{2} \dim O_\nu \qquad (1.11)$$

It is then in our example easy to see, from (1.4) and (1.8), that one may choose the real polarisations in all cases as follows

$$\begin{aligned} \ell_\nu &= \mathbb{R}\cdot I + \sum_{i=2}^{d_\nu} \sum_\alpha \mathbb{R}\cdot(t_i)^\alpha \Pi_\alpha \qquad &&, \text{ if } r \neq 0 \\ \ell_\nu &= \underline{n} &&, \text{ if } r = 0 \end{aligned} \qquad (1.12)$$

where the translations t_i belong to a maximal subgroup S^c_F of U^c_F satisfying the condition

$$(t_i)^\mu (t_{i'})^\nu\ F^{(o)}_{\mu\nu} = 0 \qquad , \forall t_i, t_{i'} \in S^c_F \quad . \qquad (1.13)$$

The coset space $\mathcal{L}/L$ may then be parametrized by a minimal set $\{t_j\}$ of translations extending S^c_F to U^c_F and the induced representations are then straightforwardly obtained from (1.6), (1.7) and (1.12) as given, by

$$V_{\nu,\underline{\ell}}(\langle f,t\rangle)\varphi(t_1) = T_{\nu,\underline{\ell}}(\langle f + f(t_1,t_2,t_3),t\rangle)\varphi(t_1 + t_3) \qquad (1.14)$$

where $t = t_2 + t_3$, $t_2 = t \cap S$, $t_3 = t - t_2 \in \mathcal{L}/L$, t_1 some element of $\mathcal{L}/L \cong U_F^c/S_F^c$ (with the usual identification coset space - coset representatives labels). Further

$$f(t_1, t_2, t_3) \overset{\mathrm{def}}{=} f(t_1, t_2 + t_3) + f(t_1 + t_3, t_2) \tag{1.15}$$

with $f(t,t')$ the factor system as given in the exponent of (1.2). Finally the functions φ in (1.14) are squared integrable measurable complex-valued functions with respect to the invariant (here Lebesgue) measure on the real vector space of dimension $m_o - \dim(S_F^c)$. Details have been dropped since the calculations are perfectly analogous to the more general case considered in [4].

Examination of (1.14) shows, as a general result, that except if $d_\nu = m_o$ (as it is of course the case whenever the field $F_{\mu\nu}^{(c)} = 0$) where these representations are the usual one-dimensional ones, we always have infinite dimensional representations for $\mathcal{L}$. This corresponds to the fact that a translational symmetry induces a shift in the momentum operator along the directions of space-time where the momentum components are not conserved quantities.

Let us note finally that the groups $\mathcal{L}$ which may appear in (1.1) are all of nilpotent length smaller or equal to 2, so that the factor systems necessary for a further induction of the representations (1.14) to the whole invariance operator groups can be directly obtained from our general result (2.21) of [4].

2. Discrete translations.

As a second basic building block of the general representations of the invariance operator groups of the equation of motion in presence of an external e.m. field, we consider the case when this group is generated by discrete translational symmetries of the field only. For simplicity, and since this case includes the other ones, we shall assume the discrete translation group U_F^d to generate a 4-dimensional lattice in space and time (physical examples of such lattices can be found in Chapter I). A basis is correspondingly denoted $b_1, b_2, \dots b_4$ so that $\forall\, a \in U_F^d$ we have

$$a = \sum_{i=1}^{4} n_i b_i \qquad , \qquad n_i \in \mathbb{Z} \qquad . \tag{2.1}$$

A reciprocal basis will be denoted $b_1^*, b_2^* \dots, b_4^*$ and is defined as is usual by

$$(b_i^* , b_j) = 2\pi\, \delta_{ij} \qquad \forall\, i,j = 1,2,3,4 \ . \tag{2.2}$$

The group $\mathcal{L}$ appears then, similarly as in (1.1) in the following exact sequence of groups

$$0 \longrightarrow U(\mathcal{F}) \longrightarrow \mathcal{L} \longrightarrow \mathbb{Z}^4 \longrightarrow 1, \qquad \omega_f, \phi \tag{2.3}$$

where ω_f is as in (1.2) and ϕ is again trivial. It is clear that we shall only consider here the case of rational fields (see Proposition 3.1 of [2]), so that $U(\mathcal{F})$, which is by definition the subgroup of $U(1)$ generated by ω_f, is a finite discrete set.

The factorization (2.3) of $\mathcal{L}$ can be used to calculate by induction the representations of $\mathcal{L}$ from the ones of $U(\mathcal{F})$. However, since the sequence is not split, projective representations of $\mathbb{Z}^4$ will be needed [6], and in fact with factor system ω_f (because ϕ is trivial) so that the problem remains the same. In order to find a more convenient factorization, we define a sequence $\mathcal{Q}_i$ of subgroups of $\mathcal{Q}$ as follows: using (2.1) we write an element $q = \langle f,a\rangle \in \mathcal{Q}$ as $q = \langle f,n_1,n_2,n_3,n_4\rangle$ and we define

$$\mathcal{Q}_i = \{ \langle f,n_1,\ldots n_i,0,0\ldots\rangle \mid (n_1,\ldots n_i) \in \mathbb{Z}^{n_i} \} \subseteq \mathcal{Q} \qquad (2.4)$$

so that obviously $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subsetneq \ldots \subseteq \mathcal{Q}$. Using now the group law of $\mathcal{Q}$ we find, with $q_i \in \mathcal{Q}_i$ $q_{i+1} \in \mathcal{Q}_{i+1}$, and using the fact that f is anti-symmetric

$$q_{i+1}q_iq_{i+1}^{-1} = \langle f,n_1,\ldots n_{i+1},0\ldots\rangle\langle f',n_1',\ldots n_i',0\ldots\rangle\langle -f,-n_1,\ldots,-n_{i+1},0\ldots\rangle$$

$$= \langle f' + 2f(\sum_{j=1}^{i+1} n_jb_j, \sum_{k=1}^{i} n_k'b_k),\ n_1'\ldots n_i',0\ldots\rangle \qquad . \qquad (2.5)$$

Using these results one can easily verify the following properties

(i) $\mathcal{Q}_i \lhd \mathcal{Q}_{i+1}$ (i.e. $\mathcal{Q}_i$ normal in $\mathcal{Q}_{i+1}$, $i = 1,2,3$)

(ii) $\mathcal{Q}_{i+1}/\mathcal{Q}_i \cong \mathbb{Z}$ (2.6)

(iii) $\mathcal{Q}_{i+1} \cong \mathcal{Q}_i \wedge_{\psi_i} \mathbb{Z}$ (i.e. semidirect).

The action ψ_i of $\mathbb{Z}$ on $\mathcal{Q}_i$ in (2.6) is in principle non trivial and can be given by

$$\psi_i(n)\langle f,n_1,\ldots,n_i,0\ldots\rangle = \langle f + 2f(n.b_{i+1}, \sum_{j=1}^{i} n_jb_j),n_1,\ldots n_i,0\ldots\rangle$$

$$= \langle f + 2n \sum_{j=1}^{i} n_jf(b_{i+1},b_j)\ ,n_1,\ldots n_i,0\ldots\rangle$$

where we have used (2.5) and the following (monomorphic) section $r : \mathbb{Z} \dashrightarrow \mathcal{Q}_{i+1}$

$$r(n) \stackrel{\text{def}}{=} \langle 0,0,\ldots,n_{i+1},0\ldots\rangle \quad , \quad \forall\, n \in \mathbb{Z}$$

These properties are illustrated in the following diagram of exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{Q}_1 \arrow[r] \arrow[d, tail] & \mathcal{Q}_2 \arrow[r] \arrow[d, tail] \arrow[dl, equal] & \mathbb{Z} \arrow[r] \arrow[d, tail] & 1, & 0,\psi_1 \\
1 \arrow[r] & \mathcal{Q}_2 \arrow[r] \arrow[d, tail] & \mathcal{Q}_3 \arrow[r] \arrow[d, tail] \arrow[dl, equal] & \mathbb{Z} \arrow[r] \arrow[d, tail] & 1, & 0,\psi_2 \\
1 \arrow[r] & \mathcal{Q}_3 \arrow[r] & \mathcal{Q} \arrow[r] & \mathbb{Z} \arrow[r] & 1, & 0,\psi_3
\end{tikzcd}
\qquad (2.7)
$$

This decomposition may now be used to calculate the representations of $\mathcal{Q}$ by inducing in steps: indeed, since all lines are split, projective representations could only be needed in the second or third step from the fact that the corresponding normal subgroups are in general not abelian. The factor systems are however factor systems on $\mathbb{Z}$ (or on subgroups of $\mathbb{Z}$) hence are necessarily trivial. The factorization (2.7) reduces thus completely the problem of the factor systems and the representations of $\mathcal{Q}$ can now be obtained by ordinary representation theory.

Let us now show more explicitly how this induction in steps works:

a) The (irreducible unitary) representations of $\mathcal{Q}_1$ are, since this group is abelian immediately found and are given, for an element $\langle f,n_1\rangle = \langle f,n_1,0,\ldots,0\rangle \in \mathcal{Q}_1$, by

$$\hat{q}_1^{\lambda,p_1}(\langle f,n_1\rangle) = \exp(i\lambda f)\exp i(2\pi n_1\cdot p_1) \qquad (2.8)$$

where $\lambda = n\cdot e/c\hbar$, $n \in \mathbb{Z}$. The physical representations (defined in (3.1) of [2] (for the relativistic, respectively in (3.1) of Chapter II for the non-relativistic problem) are then given by $n = -1$. We shall therefore in the sequel only consider this case. Further in (2.8)

$$0 \leq p_1 < 1 \qquad . \qquad (2.9)$$

Note here that p_1 is, in the case of crystal fields, nothing else than the component of the pseudomomentum along l_1^*.
The action of $\mathbb{Z}$ on these representations is then obtained from the canonical map $\hat{\Psi}_1$ with

$$(\hat{\Psi}_1(n_2)\hat{q}_1)(<f,n_1>) \overset{\text{def}}{=} \hat{q}_1(\psi_1(n_2)^{-1}<f,n_1>) \quad .$$

One finds straightforwardly that this implies

$$\hat{\Psi}_1(n_2)\, \hat{q}_1^{\lambda,p_1} = \hat{q}_1^{\lambda,p_1+n_2(\frac{\lambda}{2\pi})\, G^{(o)}_{21}} \tag{2.10}$$

where

$$G^{(o)}_{ij} = S^{\mu}_{i} S^{\nu}_{j} F^{(o)}_{\mu\nu} = -G^{(o)}_{ji} \;, \quad i,j = 1,..4 \tag{2.11}$$

is the c.u. field in the b_i-basis coordinates, S being the corresponding basis transformation matrix, so that, by definition, we have simply

$$G^{(o)}_{ij} = 2\, f(b_i,b_j) \tag{2.12}$$

If $2\lambda.f(b_1,b_2) = 0$ (mod. 2π) then $\mathcal{L}_2$ is abelian and one may directly start with its representations as above so that one induction step simply drops. We may thus suppose that it is not the case. Since $U(\mathcal{F})$ is a finite set, there exists however a (smallest) positive integer N_2 such that

$$N_2 \;\; 2\lambda f(b_1,b_2) = 0 \;\; (\text{mod } 2\pi) \quad . \tag{2.13}$$

The stability factor group $H_2 \subseteq \mathbb{Z}$ of $\hat{q}_1$ is then given by (through its imbedding in $\mathcal{L}_2$)

$$H_2 = \{ <0,0,\mathbb{Z}\cdot N_2> \} \subseteq \mathcal{L}_2 \tag{2.14}$$

and the orbits in the character space of $\mathcal{Z}_1$ are finite (of order N_2), each orbit being then uniquely characterized by a real number p_1 with

$$0 \leqslant p_1 < 1/N_2 \quad . \tag{2.15}$$

The coset space $\mathcal{Z}_2/(\mathcal{Z}_1 \circledS H_2)$ is then also finite and of order N_2 and can be parametrized by the integers $k_i \in \{0,1\ldots,N_2-1\}$ characterising the corresponding translations $k_i b_2$ (again after identification coset space-coset representatives labels). The induced representations $(\hat{q}_1 \uparrow \mathcal{Z}_2)$ act then on the Hilbert space of square summable functions $g(k_i)$ having values in the representation space of $\hat{q}_1$, and are defined as follows. We first construct an irreducible representation Γ of the little group $\mathcal{Z}_1 \circledS H_2$ by

$$\Gamma(<f,n_1,mN_2>) = \hat{q}_1(<f,n_1>) \otimes \hat{h}_2(m) \tag{2.16}$$

with $m \in \mathbb{Z}$ and $\hat{h}_2$ is an ordinary (since $\mathcal{Z}_1$ is abelian and the first line of (2.7) is split) representation of H_2, hence is given by

$$\hat{h}_2(m) = \exp i\, 2\pi\, m\, p_2 \tag{2.17}$$

with some real parameter p_2 in the set

$$0 \leqslant p_2 < 1 \quad . \tag{2.18}$$

The representations (2.16) can then be induced to $\mathcal{Z}_2$ and give the following complete set of irreducible representations of $\mathcal{Z}_2$:

$$\hat{q}_2^{\lambda,p_1,p_2}(<f,n_1,n_2>)g(k_i) \overset{\text{def}}{=} \Gamma(<0,0,k_i><f,n_1,n_2><0,0,-k_j>)g(k_j) \tag{2.19}$$

where k_j is such that $k_i-k_j+n_2 = m.N_2$, for some $m \in \mathbb{Z}$.

Inserting this condition in (2.19) one finds finally

$$\hat{q}_2^{\lambda,p_1p_2}(\langle f,n_1,n_2\rangle)g(k_i) = \Gamma(\langle f+(2k_i+k)n_1f(b_2,b_1),n_1,m'\rangle)g(k_i+k) \quad (2.20)$$

where k is the coset representative of n_2 ($n_2 = mN_2+k$, $0\leq k<N_2$) and

$$m' = \begin{cases} m & \text{if } k_i+k < N_2 \\ m+1 & \text{if } k_i+k \geq N_2 \end{cases} \quad (2.21)$$

These representations are all finite dimensional (dimension N_2), the general matrix form being as follows

$$\hat{q}_2(\langle f,n_1,n_2\rangle) = \left(\begin{array}{c|c} \bigcirc & \Gamma(\langle f',n_1,m\rangle)\cdot \mathbb{1}_{N_2-k} \\ \hline \Gamma(\langle f',n_1,m+1\rangle)\ \mathbb{1}_k & \bigcirc \end{array}\right) \quad (2.22)$$

where f' is a shorthand notation for the expression in (2.20) and $\mathbb{1}_d$ is the unit matrix in d dimensions.

b) In a next step we consider the second line in (2.7) and induce the just found representations to $\mathcal{L}_3$. The essential point of the procedure is then the fact that the representations in (2.20) are (in general) not one-dimensional. Hence that the problem is a little more difficult than in the usual case since explicit intertwining matrices need to be given. As we shall see, however, this difficulty can easily be solved in our case.

The action $\hat{\psi}_2(n_3)$ of $\mathbb{Z}$ on the dual of $\mathcal{L}_2$ is calculated similarly as in (2.10) and gives, with (2.20)

$$\hat{\psi}_2(n_3)\ \hat{q}_2^{\lambda,p_1,p_2} = \hat{q}_2^{\lambda,p_1+n_3(\frac{\lambda}{2\pi})G_{31}^{(o)},p_2+n_3(\frac{\lambda}{2\pi})G_{32}^{(o)}} \quad (2.23)$$

hence the stability factor group, defined similarly as in (2.14) is given by

$$H_3 = \{<0,0,0,\mathbb{Z} N_3>\} \subseteq \mathcal{Q}_3 \qquad (2.24)$$

where N_3 is the smallest integer such that

$$N_3(\frac{\lambda}{2\pi})\, G_{31}^{(o)} = \frac{\ell}{N_2}\,,\ \ell \in \mathbb{Z} \text{ and } N_3 .\ (\frac{\lambda}{2\pi})\, G_{32}^{(o)} \in \mathbb{Z}\ . \qquad (2.25)$$

The intertwining matrix $S(N_3)$ corresponding to the generator of H_3 and which is defined by the condition

$$\mathfrak{U}_2(N_3)\ \hat{q}_2^{\lambda,p_1,p_2} = S(N_3)^{-1}\ \hat{q}_2^{\lambda,p_1,p_2}\ S(N_3)$$

is then easily found to be, up to an uninteresting non zero constant, given by the displacement matrix

$$S(N_3) = S_\ell = \begin{pmatrix} 0 & \mathbb{1}_{N_2-\ell} \\ \mathbb{1}_\ell & 0 \end{pmatrix} \qquad (2.26)$$

where ℓ is defined by (2.25) modulo N_2.

Since the matrices (2.26) satisfy, $\forall\, \ell,\ \ell' \in \mathbb{Z}$ (mod N_2)

$$S_\ell \cdot S_{\ell'} = S_{\ell+\ell'} \qquad (2.27)$$

the factor system needed in the induction procedure is not only equivalent but equal to 1. The induced representations $(\hat{q}_2 \uparrow \mathcal{Q}_3)$ are then obtained similarly as in (2.20) with g a squared summable vector-valued function on the discrete (N_3 points) orbit of each representation (2.20) and with values on the carrier space of $\hat{q}_2$. The corresponding representation Γ of the little group $\mathcal{Q}_2 \circledS H_3$ is thereby given by

$$\Gamma(<f,n_1,n_2,m.N_3>) = \hat{q}_2(<f,n_1,n_2>).S_{m.\ell} \otimes \hat{h}_3(m) \qquad (2.28)$$

with $S_{m.\ell}$ as in (2.26), $m \in \mathbb{Z}$ and $\hat{h}_3(m)$ a (one-dimensional) irreducible representation of H_3.
The last step (induction from $\mathcal{Q}_3$) is then perfectly similar, with then as intertwining matrices tensor products of matrices of the form (2.26). Details are tedious but straightforward and are therefore omitted. More interesting at this point is to analyse the consequences of the special form of these representations as compared to the ordinary situation (as without c.u. field) for the solutions of the equations of motion. In particular we want to show how the well known Bloch-ansatz has to be modified in presence of an external e.m. field with non-trivial factor system f. For simplicity we consider the action of a one-dimensional subgroup of the discrete translational symmetries of the field, the problem being similar in the other dimensions.

If $F^{(o)} = 0$, the physical representations are defined on square integrable functions on the real space via the substitution operator. Let thus b be the generator of this translation group, T_b the corresponding substitution operator and $\psi(x)$ a solution of the equation of motion. We may choose $\psi(x)$ as basis element of some irreducible representation of the group generated by T_b, i.e. such that it satisfies

$$T_b\, \psi(x) = \psi(x-b) = e^{ipb}\, \psi(x) \qquad (2.29)$$

setting now $\psi(x) = e^{-ipx}\, u_p(x)$ one obtains the usual condition

$$u_p(x) = u_p(x-b) \qquad (2.30)$$

so that the function $u_p(x)$ can be expanded as a Fourier series

$$u_p(x) = \sum_K \hat{u}_{pK} \exp(iKx) \tag{2.31}$$

where the sum goes over all K which satisfy the condition $Kb = 2n\pi$ for some $n \in \mathbb{Z}$. The problem of the (differential) equation of motion is transformed in this way in an easier algebraic one for the coefficients $\hat{u}_{pK}$ of (2.31), as is well known.

Let us now consider the situation where an external field is present, such that $\omega_f(b,b') \neq 1$ for two generators b and b' of the discrete translational symmetry group of the field. Choosing similarly as above the solutions $\psi(x)$ as basis functions of an irreducible representation of the operator group generated by these two translations, and identifying b with b_2 and b' with b_1 in (2.1) and (2.20) we have, in place of (2.29)

$$V_b \, \psi_s(x) \equiv U_{\chi_b} T_b \, \psi_s(x) = e^{ipb} \psi_{s'}(x-b) \tag{2.32}$$

where U_{χ_b} is the corresponding (compensating) gauge transformation (see e.g. [2] part 3) and s, s' are additional integer labels (defined,as we saw,modulo some integer N), s' being determined by b as in (2.20).

We may now set again $\psi_s(x) = e^{-ipx} u_{sp}(x)$ and we find in place of (2.30) the more general relation

$$u_{ps}(x-b) = u_{ps'}(x) \quad . \tag{2.33}$$

Since s, s' are members of a finite set (and this follows of course from the fact that the field has been chosen rational) we have necessarily

$$u_{ps}(x-N \cdot b) = u_{ps}(x) \tag{2.34}$$

so that $u_{ps}(x)$ may also be expanded as a Fourier series

$$u_{ps}(x) = \sum_{K'}' \hat{u}_{psK'} \exp (iK'x) \tag{2.35}$$

with K' satisfying now $K'.Nb = 2n\pi$, $n \in \mathbb{Z}$. Together with (2.33) we find then the additional relations between the Fourier coefficients of the functions u_{ps} and $u_{ps'}$

$$\hat{u}_{ps'K} = \exp (-iKb)\, \hat{u}_{psK} \tag{2.36}$$

and again the problem may be treated algebraically.

The expansion (2.35) corresponds in fact, for the case of a crystal lattice in a c.u. magnetic field, to an expansion in a mini-Brillouin zone (the so-called magnetic reciprocal unit cell). The result is however more general since any space-time lattice can be treated in this way, and in any external field. On the other side the condition (2.36) shows an essential difference with ordinary superstructure since it corresponds to an interdependence of Bloch functions of the type (2.35) related to p-vectors whose difference is a reciprocal superlattice vector within the old Brillouin zone.

Let us remark finally that the dimensions of the representations we have analyzed in this part are quite pathological, even if the field is supposed to be rational, since an infinitesimal (rational) change of the value of the field may induce arbitrary large changes in the integers N_i which determine these dimensions. If the field is not rational, the condition (2.34) does of course no longer hold so that a generalized Bloch ansatz is in this case not possible, and this is not surprising since, as we have shown previously, the operator group $\mathcal{Q}$ is then not of Type I.

References

[1] A. Janner and T. Janssen, Physica 60, 292 (1972)

[2] N. Giovannini, Chapter I of the present treatise

[3] A. Kirillov, Uspekhi Mat. Nauk. 17, 57 (1962)

[4] N. Giovannini, Appendix 5A of the present treatise

[5] A. Guichardet, Séminaire Bourbaki 1962/63, no. 249

[6] G.W. Mackey, Acta Math. 99, 267 (1958) and ref. in there.

COVARIANCE IN THE PRESENCE OF EXTERNAL ELECTROMAGNETIC FIELDS

Introduction

The group theoretical definition of an elementary particle as a continuous projective irreducible representation of the Poincaré group IO(3,1) is a well known and successful one, since the celebrated paper of Wigner [1]. In this frame, the Poincaré group plays the role of the covariance group of special relativity, i.e. embodies the basic postulate of the theory that physical results should be left invariant under a space-time change of reference frame relating by definition two inertial systems. The elementary particles can then be characterized by the values of their spin and their squared mass and it is also possible, using group theoretical methods, to tackle successfully the description of the time evolution of the states representing such particles, i.e. to relate covariant equations of motion to representations of this group.

Let us now introduce an electromagnetic field F and consider a charged particle moving in it. As it has often been shown in the past in many situations the so-called "external" approximation (where the influence of the particle on the field is neglected), is a very good and useful one. For that purpose, the usual procedure is to introduce the field F via a potential A in the free equation of motion by minimal coupling, the momentum operators P_μ being replaced by $\Pi_\mu = P_\mu - \frac{e}{c} A_\mu(x)$ (e the charge of the particle). Again the results of the theory are very satisfying, at least for spin 0 and $\frac{1}{2}$ particles (we shall discuss later [2] the serious difficulties encountered in the higher spin cases [3,4]).

Physics however depend on fields and not on potentials and should thus be *invariant with respect to any change in field components* due to a covariant change of reference frame. Introducing a potential in the formalism induces in fact some arbitrariness: first, to a given field does not correspond one potential but a whole class of them, hence not one equation of motion but an infinite (physically undistinguishable) class of them. Second, and as a consequence of this, there is a certain choice left for the definition of the transformation law of the potential that we have introduced. Therefore one may require the potential to transform as a covariant (time pseudo-) vector field under the Poincaré group but not the modified equation of motion to be Poincaré covariant: Poincaré covariance has then only to hold for the (infinite) class of equations corresponding to a given field and not for each element of that class separately. This implies incidentally also that arbitrary gauge transformations are covariant transformations for this class, too, and this gives rise to interesting but infinite dimensional groups which are quite difficult to handle [5,6]). This is of course not new and a natural alternative would be to develop a formalism without introducing a potential, but based only on fields, as it has been done successfully for Dirac particles [7]. A generalization of such a formalism is however not known.

The reason for this is the probably too fundamental role of gauge invariance.

In the present work we choose a way inbetween, using the advantages of these two formalisms, i.e. we use potentials, but we get rid of the arbitrariness described above by fixing in some convenient way the gauge, for any given field. In this way it is possible to build up group theoretically a theory of elementary particles in interaction with an external field, by the construction first of the relevant covariance group, independently of any equation of motion, and by the analysis then of the representations of this group.

This chapter will be organized as follows: in the first part we define more precisely what we mean by covariance. Then, after having made also more precise in part two the class of fields we consider, we define covariant transformations as acting on the potential space of a given field. This will be done, following our philosophy, with the help of the quite natural concept of compensating gauges together with a (fixed) chosen map $\pi: F \longmapsto A$ which applies any e.m. field F in an uniquely determined potential A. The result is shown to be essentially independent of the particular choice for π. In this way we construct covariance operator groups that may however, by construction, depend on the field we started with. We get rid of this dependence in the fourth part, by expliciting the general covariance group, valid for any field, and by making also explicit its relationship with the concept of covariant equation of motion in the presence of an external field. The representations of this group and their physical interpretation and consequences are the subject of a subsequent paper [2].

1. Covariance (definition).

Because the term covariance is quite a much used one (and unfortunately not always with the same meaning), let us first briefly define what we mean more precisely by covariance and covariance group. Let therefore α be an element of some parameter space $\mathbf{X}$, $\psi \in \mathcal{H}(\mathbf{X})$ some (separable) Hilbert space of functions on $\mathbf{X}$ and

$$O(\alpha)\ \psi(\alpha) = 0 \quad , \qquad \forall\, \alpha \in \mathbf{X} \tag{1.1}$$

some (scalar) wave equation on $\mathcal{H}(\mathbf{X})$ (e.g. an equation of motion). Let now g be an invertible map from $\mathbf{X}$ into itself with $g\alpha = \alpha'$. Then g is called a <u>covariant</u> transformation for O if and only if there exists a unitary (or antiunitary) operator V_g on $\mathcal{H}(\mathbf{X})$ such that

$$O'(g\alpha) \equiv V_g\ O(g\alpha)\ V_g^{-1} = O(\alpha) \tag{1.2}$$

implying of course that in the new frame, with $\psi'(\alpha) \equiv (V_g\psi)(\alpha)$ we have, for all ψ satisfying (1.1)

$$O'(\alpha)\ \psi'(\alpha) = 0 \quad . \tag{1.1$'$}$$

Note that in our case $\mathbf{X}$ will depend not only on space-time coordinates but also on a potential of the external field considered. The condition of covariance (1.2) is then as made more explicit in (4.11).

The above definition may be of course generalized to the case where $\psi(\alpha)$ is a n-components wave function and $O(\alpha)$ is in n x n matrix form: g will be then called a covariant transformation for $\mathbf{O}$ if and only if there exist an unitary (or antiunitary) operator V_g in $\mathcal{H}(\mathbf{X})$ and a non singular n x n matrix Λ_g which satisfy the condition

$$O'(g\alpha) \equiv V_g \, O(g\alpha) \, V_g^{-1} = \Lambda_g \, O(\alpha) \, \Lambda_g^{-1} \tag{1.2)'}$$

implying the same equation (1.1)' as before with similarly $\psi'(\alpha)$ given by $(V_g\psi)(\alpha)$.

One may verify that this definition implies that the elements g which are covariant transformations form a group G, the covariance group. Furthermore, if the set of all functions $\{(V_g\psi)\}$ in $\mathcal{H}(X)$ does not contain proper invariant subspaces, it follows, also from the definition and from Schur's Lemma, that the set $\{V_g | \forall\, g \in g\}$ satisfies the extra condition

$$V_g \cdot V_{g'} = \omega(g, g') \, V_{gg'} \tag{1.3}$$

where $\omega(g, g')$ is some complex number of modulus one, so that, taking into account the associativity properties of operators on Hilbert spaces and of matrices, the set $\{V_g\}$ forms an irreducible projective representation of G with carrier space $\mathcal{H}(X)$.

We define further g to be an <u>invariant</u> transformation of O, or a symmetry for O, if g is a covariant transformation with in addition

$$O'(g\alpha) = O(g\alpha) \quad . \tag{1.4}$$

It is important to note here that for G the Poincaré group and O the operator of an ordinary (free) equation of motion, the two concepts coincide. This will however no longer be so in our case covariance embodies the property of equivalence of reference systems (and there will thus be elements related to <u>every</u> Poincaré transformation) whereas invariance will be generated by the covariant transformations which in addition leave the external field invariant. As we shall see, in both cases it will be in general necessary to combine in a non-trivial way gauge and Poincaré transformations.

The concept of covariance is thus closely related to the equation of motion we look at. In this chapter however we shall proceed in the other direction: we shall construct first, on a physical basis, the general covariance group for a charged particle in the presence of an external e.m. field and then extract from it information about the covariant equations of motion and their solutions.

2. External e.m. Fields.

Because they will play quite an important role with respect to our space **X**, let us make first mathematically more precise what we consider as possible candidates for an external e.m. field: an external e.m.field will be a continuous differentiable cross-section of the antisymmetric product of the cotangent spaces $T^*_x \wedge T^*_x$ at each point of the Minkovski space $M(4)$, i.e. a continuous differentiable map from $M(4)$ into the space of the real antisymmetric (time-pseudo-) tensors of rank two $F_{\mu\nu}$ with the following three properties

(i) it satisfies the Maxwell equations

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$

$$F_{\mu\nu}{}^{,\nu} = \frac{4\pi}{c} j_\mu \qquad \forall \mu,\nu = 0,1,2,3 \quad . \tag{2.1}$$

Because we make, from the physical point of view, no further restriction on the 4-current j_μ the second set of equations can and will be considered as a definition of this 4-current.

(ii) it transforms under an element $g = (a,\Lambda)$ [1] of the Poincaré group IO(3,1) according to

$$(gF)_{\mu\nu}(x) = \varepsilon(g)\, \Lambda_\mu^\rho\, \Lambda_\nu^\sigma\, F_{\rho\sigma}(\Lambda^{-1}(x-a)) \qquad (2.2)$$

with $\varepsilon(g) = \text{sign}\,(\Lambda_o^o)$.
This law defines obviously, by the condition

$$g \in G_F \quad \text{if and only if} \quad (gF)_{\mu\nu}(x) = F_{\mu\nu}(x),\ \forall x \in M(4) \qquad (2.3)$$

a subgroup G_F of IO(3,1), the symmetry group of the field F.

(iii) it can be written as an integral

$$F_{\mu\nu}(x) = \int d^4k\, \hat{F}_{\mu\nu}(k)\, \exp\,(ikx) \qquad (2.4)$$

with $k \in M^*(4)$, the dual of the Minkovski space, i.e. we consider any e.m. field for which the integral

$$\hat{F}_{\mu\nu}(k) = \left(\frac{1}{2\pi}\right)^4 \int d^4x\, F_{\mu\nu}(x)\, \exp\,(-ikx) \qquad (2.5)$$

exists, in the sense of generalized functions.

---

1) We adopt the notation $g = (a,\Lambda) \in IO(3,1)$, with $(a,\Lambda)x = \Lambda x + a$ $x \in M(4)$, a a 4-translation, $\Lambda \in O(3,1)$, an element of the homogeneous Lorentz group, $(\Lambda x)_\mu = \Lambda_\mu^\nu\, x_\nu$ for covariant and $(\Lambda x)^\mu = (\Lambda^{-1})_\nu^\mu\, x^\nu$ for contravariant vector components. In this notation the product in IO(3,1) reads then $(a,\Lambda)(a',\Lambda') = (a + \Lambda a',\ \Lambda\Lambda')$.

3. Invariant and covariant transformations acting on e.m. potentials.

a) Invariant transformations.

As explained in the introduction, we now consider for a given $F_{\mu\nu}(x)$, as is usual, and in accordance with our philosophy, a potential $A_\mu(x)$ as a real 4-vector field over M(4) satisfying

$$A_{\nu,\mu}(x) - A_{\mu,\nu}(x) = F_{\mu\nu}(x) \qquad (3.1)$$

and transforming under the Poincaré group, correspondingly to (2.2), as

$$(gA)_\mu(x) = \varepsilon(g)\, \Lambda_\mu^\nu\, A_\nu(\Lambda^{-1}(x-a)) \quad . \qquad (3.2)$$

It is clear that all $g \in IO(3,1)$ satisfying the equation

$$(gA)_\mu(x) = A_\mu(x) \qquad (3.3)$$

will satisfy the corresponding equation (2.3) for the e.m. field, too. But, as is well known, the contrary is in general not true, as for instance in the simplest case of constant uniform e.m. fields where it is not possible to find a potential having the same translational symmetry group as the field. Not only in this case, however, do the space-time transformations leaving the potential in (3.1) invariant form necessarily a smaller group than the symmetry group of the field, for any choice of gauge: another example is the electromagnetic field of a plane wave as shown by Janner and Janssen [8]. It will by the way be also a consequence of the present work to see exactly for which fields this will in fact necessarily be the case.

As mentioned before, dealing with potentials in place of fields, we have however introduced a certain arbitrariness, because potentials are not physical observables as fields are. The symmetry of the field and hence of the physical system has thus to be restored in some way. This is possible, using the quite natural concept of compensating gauge a symmetry of the field can, acting on a corresponding potential generate a new potential that can only differ from the previous one by a gauge transformation (both potentials being related to the same field). One may thus combine gauge and space-time transformations in such a way that to each symmetry of the field does correspond such a coupled transformation leaving the potential invariant and hence an operator commuting with the (potential-dependent) equations of motion The resulting situation has been analyzed group theoretically on a general basis by Janner and Janssen [9] and it has been shown in some explicit examples (see e.g. [10-12]) how drastically the structure of these operator groups can then be modified with respect to the original space-time subgroups structure.

These operator groups are then also, in general, not subgroups of the usual space-time transformation groups corresponding to the equation of motion considered. Invariant transformations, (relating identical physical systems) are however necessarily particular cases of covariant transformations (relating equivalent physical systems), as we mentioned in section 1. It is in fact on this basis that we shall now extend the just sketched treatment for covariant transformations as well. The above invariance operator groups will then appear in a natural way as (ordinary) representations of subgroups of the covariance group we shall obtain.

b) Covariant transformations.

Because we ask that, for a given external e.m. field F, there should be a covariant transformation relating any two reference frames that are mapped on each other by a Poincaré transformation, we now consider the case where g runs over the whole Poincaré group, and construct in a first step a group of transformations on the space of potentials as follows.

We first consider an arbitrary but fixed external field F and assume we have chosen some (convenient) fixed (linear) map π from the e.m. field tensors to the space of potentials

$$\pi : F \longmapsto A \quad . \tag{3.4}$$

In order to get rid of the arbitrariness introduced together with the potential we now want to construct a set $\{g^* \mid \forall g \in IO(3,1)\}$ of transformations combining gauge and Poincaré transformations and acting on the space of the potentials in such a way that the diagram

$$\begin{tikzcd} F \arrow[r, mapsto, "\pi"] \arrow[d, "g"] & A \arrow[d, "g^*"] \\ (gF) \arrow[r, mapsto] & \pi(gF) \end{tikzcd} \tag{3.5}$$

is commutative, i.e. such that

$$g^*(\pi F) = \pi(gF) \qquad \forall\, g \in IO(3,1) \quad . \tag{3.6}$$

Because of the transformation laws (2.2) and (3.2) for fields and potentials both $\pi(gF)$ and $g(\pi F)$ correspond to the same field so that we have, in general,

$$(g(\pi F))(x) = (\pi(gF))(x) + \partial\chi_g(\pi(gF),x) \tag{3.7}$$

for some gauge function $\chi_g(\pi(gF),x)$ which may depend on x, on g, on the field F and on the chosen map π. This gauge function is then fixed by (3.7) uniquely up to a constant. We may now combine, analogously as in the invariance case, any Poincaré transformation with a gauge transformation in such a way that the transformed potential is kept in the same fixed gauge as defined by the map π. We thus define, for any $g \in IO(3,1)$ a pair $\{\chi_g, g\} \overset{def}{=} g^*$, whose action on πF is given by

$$\{\chi_g, g\}\ (\pi F) \overset{def}{=} g(\pi F) - \partial\chi_g\ (\pi(gF), x) \qquad , \qquad (3.8)(i)$$

so that by construction we have, as required

$$\{\chi_g, g\}\ (\pi F) = \pi(gF) \qquad . \qquad (3.8)(ii)$$

The transformations defined by (3.8) do not however, in general, form a group, as it is not necessarily possible to choose the arbitrary constants in (3.7) in such a way that the gauge function associated with the product of two Poincaré transformations is the gauge function resulting from the product of the corresponding two pairs.

It is however possible to imbed them in a larger group $Q^*_{\pi F}$ combining constant gauge functions with Poincaré transformations, and which we now want to make more explicit.

Using first (3.2) and the fact that the operator ∂ transforms covariantly under the Poincaré group, the action of an element $g_1 \in IO(3,1)$ on the gauge functions in (3.7) may be consistently defined as follows, $\forall\, g_1, g_2 \in IO(3,1)$

$$\psi(g_1)\ \chi_{g_2}(\pi(g_2F), x) \overset{def}{=} \varepsilon(g_1)\ \chi_{g_2}(\pi(g_2F), g_1^{-1}x) \quad . \quad (3.9)$$

Using this relation and letting g_1 act on both sides of (3.7) one then obtains

$$\partial\psi(g_1)\chi_{g_2}(\pi(g_2F),x) - \partial\chi_{g_1g_2}(\pi(g_1g_2F),g_1^{-1}x) + \partial\chi_{g_1}(\pi(g_1g_2F),g_1^{-1}x) = 0 \, .$$
$$(3.10)$$

The relation (3.10) can be integrated and one finds (after a shift in the field variable)

$$\chi_{g_1}(\pi F,x) + \psi(g_1)\chi_{g_2}(\pi(g_1^{-1}F),x) - \chi_{g_1 g_2}(\pi F,x) \in \mathbb{R} \quad . \tag{3.11}$$

This expression does thus not depend on x but may of course depend on πF and on the space-time transformations g_1 and g_2. We shall denote this function in the sequel by $(f^*(g_1,g_2))(\pi F)$. Rewriting the second member of (3.11) as

$$\psi(g_1)\chi_{g_2}(\pi(g_1^{-1}F),x) \overset{\text{def}}{=} \zeta(g_1)\chi_{g_2}(\pi F,x) \tag{3.12}$$

are finally obtains for (3.11)

$$(f^*(g_1,g_2))(\pi F) = \chi_{g_1}(\pi F,x) + \zeta(g_1)\chi_{g_2}(\pi F,x) - \chi_{g_1 g_2}(\pi F,x). \tag{3.13}$$

Denoting now by $\Phi_{\pi F}$ the additive group of real functions of πF generated by the functions in (3.13), $\forall\, g_1,g_2 \in IO(3,1)$, and with product

$$(\varphi_1 + \varphi_2)(\pi F) \overset{\text{def}}{=} \varphi_1(\pi F) + \varphi_2(\pi F) \;, \forall\, \varphi_1,\varphi_2 \in \Phi_{\pi F}, \forall F \in O_F \tag{3.14}$$

where O_F denotes the orbit of F under the Poincaré group, one finds that the group $Q^*_{\pi F}$ which is <u>generated</u> by the transformations in (3.8) has the following structure:

<u>Proposition 3.1.</u> $Q^*_{\pi F}$ is an extension of $\Phi_{\pi F}$ by $IO(3,1)$ with product rule, $\forall\, \varphi_1, \varphi_2 \in \Phi_{\pi F}$ and $\forall\, g_1, g_2 \in IO(3,1)$

$$(\varphi_1, g_1)(\varphi_2, g_2) = (\varphi_1 + \zeta(g_1)\,\varphi_2 + f^*(g_1,g_2),\; g_1 g_2) \tag{3.15}$$

where $f^*(g_1, g_2)$ is given by (3.13) and $\zeta(g_1)$ is defined, from (3.12) and (3.13), by

$$(\zeta(g_1)\varphi)(\pi F) \overset{\text{def}}{=} \varepsilon(g_1)\varphi(\pi(g_1^{-1}F)) \qquad . \tag{3.16}$$

the possible sign $\varepsilon(g_1)$ in (3.10) being as in (2.2) and (3.9). These properties can be summarized in the following exact sequence of groups:

$$0 \longrightarrow \Phi_{\pi F} \longrightarrow Q^*_{\pi F} \longrightarrow IO(3,1) \longrightarrow 1, \qquad f^*, \zeta \quad .(3.17)$$

<u>Proof:</u> That $\Phi_{\pi F}$ is normal in $Q^*_{\pi F}$ and that $Q^*_{\pi F}/\Phi_{\pi F} \cong IO(3,1)$ follow from the definitions. We just have to verify that $Q^*_{\pi F}$ is indeed the group generated by the transformations g^* in (3.8) and that the functions f^* in (3.13) do satisfy the factor system conditions.

For the former, defining the imbedding of g^* in $Q^*_{\pi F}$, $\forall\, g \in IO(3,1)$, by the following map r

$$r \cdot g^* \equiv r\{\chi_g, g\} \overset{\text{def}}{=} (0,g) \in Q^*_{\pi F} \tag{3.18}$$

the verification is straightforward. Note that the map $g \longmapsto r(g^*)$ is of course nothing else than a section for (3.17). Conversely the action of $Q^*_{\pi F}$ in (3.17) on the potential subspace defined by π, $\forall\, F \in O_F$, is given by the mapping σ defined by

$$(\sigma(\varphi,g))(\pi F) \overset{\text{def}}{=} \{\chi_g + \varphi, g\}(\pi F) = \pi(gF) \tag{3.19}$$

where we have used (3.7), (3.8) and the definition of $\Phi_{\pi F}$.

Finally, using the definitions (3.13) and (3.16), it is tedious but straightforward to verify that

$$f^* : IO(3,1) \times IO(3,1) \longrightarrow \Phi_{\pi F} \tag{3.20}$$

satisfies, $\forall\, F \in O_F$ and $\forall\, g_1, g_2, g_3 \in IO(3,1)$ the identity

$$f^*(g_1, g_2 g_3) + \zeta(g_1)\, f^*(g_2, g_3) \;=\; f^*(g_1, g_2) + f^*(g_1 g_2, g_3) \quad (3.21)$$

so that f^* is effectively a factor system.

We shall later on specify the explicit structure of this group, and especially the form of the functions in $\Phi_{\pi F}$. Before we do so, we first show that our definition is consistent, in the sense that $Q^*_{\pi F}$ does not, as an abstract group, depend on the particular choice of gauge we have made by fixing π, nor on the choice of a particular reference frame.

Proposition 3.2.

$$Q^*_{\pi F} \;\cong\; Q^*_{\pi(gF)} \;\cong\; Q^*_{\pi F + \partial\xi(\pi F, x)} \qquad (3.22)$$

$\forall\, g \in IO(3,1)$ and for all differentiable (gauge) functions $\xi(\pi F, x)$ on space-time.

Proof: The first relation follows from the fact that a change of reference frame induces a conjugation in the r.h. side of (3.17) giving rise to an isomorphic extension. Further, let π' be a new choice of gauge (which may also be different for different $F \in O_F$), i.e. let

$$A'(x) \;=\; (\pi' F)(x) \;=\; (\pi F)(x) + \partial\xi(\pi F, x)$$

for some gauge functions $\xi(\pi F, x)$. We then have

$$(\pi(gF) - g(\pi F))(x) \;=\; \partial\chi_g(\pi(gF), x)$$

$$\begin{aligned}(\pi'(gF) - g(\pi' F))(x) \;&=\; (\pi(gF) - g(\pi F))(x) \\ &\quad + \partial\xi(\pi(gF), x) - \psi(g)\partial\xi(\pi F, x)\end{aligned} \qquad (3.23)$$

where we have used the linearity of the action of the Poincaré group on potentials. It follows from (3.23), ∂ transforming covariantly under IO(3,1), that

$$\chi_g(\pi'(gF),x) - \chi_g(\pi(gF),x) = (1 - \zeta(g))\ \xi(\pi(gF),x)$$

up to an arbitrary (inessential) constant function of πF.

Inserting this relation in (3.13) one then obtains

$$(f^*(g_1,g_2))(\pi F) - (f^*(g_1,g_2))(\pi' F) = 0\ ,\forall\ g_1 g_2 \in IO(3,1)$$

so that both factor sets describe equivalent thus a fortiori isomorphic extensions. This completes the proof.

Let us now turn to the explicit calculation of the structure of our group. As a consequence of Proposition 3.2 we may choose some convenient gauge for the potential, i.e. some convenient map π. Noting therefore also that for various kinds of fields there exist more natural choices we do it as follows: we split the spectrum S of the field, i.e. the set of all k in (2.5) for which $\hat{F}_{\mu\nu}(k) \neq 0$ in three disconnected parts

$$\begin{aligned} S &= S^{(o)} \cup S^{(r)} \cup S^{(j)} \\ S^{(\alpha)} \cap S^{(\beta)} &= \emptyset\ , \qquad \alpha \neq \beta,\ \alpha,\beta \in o,r,j \end{aligned} \tag{3.24}$$

with

$$\begin{aligned} S^{(o)} &= \{k \in S \mid k = 0\} \\ S^{(r)} &= \{k \in S \mid k^2 = 0,\ \ k \neq 0\} \\ S^{(j)} &= \{k \in S \mid k^2 \neq 0\} \end{aligned} \tag{3.25}$$

Note that this decomposition is Poincaré invariant. The field F splits correspondingly to (3.25) in three (independent) e.m. fields:

$$F_{\mu\nu}(x) = F^{(o)}_{\mu\nu} + F^{(r)}_{\mu\nu}(x) + F^{(j)}_{\mu\nu}(x) \tag{3.26}$$

where $F^{(o)}_{\mu\nu}$ will be assumed for the sake of this work to consist in a constant uniform field only, $F^{(r)}_{\mu\nu}$ is a radiation field, $F^{(j)}_{\mu\nu}$ a "current" field. All these fields are obviously defined for $\alpha = o,r,j$, by

$$F^{(\alpha)}_{\mu\nu}(x) = \int d^4k \, \hat{F}^{(\alpha)}_{\mu\nu}(k) \exp (ikx) \tag{3.27}$$

with

$$\hat{F}^{(\alpha)}_{\mu\nu}(k) = \begin{cases} \hat{F}_{\mu\nu}(k) & k \in S^{(\alpha)} \\ 0 & \text{else} \end{cases} \quad . \tag{3.28}$$

As each of these fields has to satisfy independently the Maxwell equations (2.1), one may choose for each of them a corresponding potential in a specific convenient gauge. We define thus a map π by its action on the various parts of (3.24)

$$(\pi F^{(\alpha)})(x) = A^{(\alpha)}(x)$$

with

$$\begin{aligned} A^{(o)}_{\mu}(x) &\overset{\text{def}}{=} \tfrac{1}{2} \, x^{\rho} \, F^{(o)}_{\rho\mu} \\ A^{(r)}_{\mu}(x) &\overset{\text{def}}{=} \int d^4k \left(\frac{\hat{F}^{(r)}_{o\mu}(k)}{i \, k_o} \right) \exp (ikx) \\ A^{(j)}_{\mu}(x) &\overset{\text{def}}{=} \int d^4k \left(\frac{k^{\rho} \hat{F}^{(j)}_{\rho\mu}(k)}{k_{\rho} k^{\rho}} \right) \exp (ikx) \end{aligned} \tag{3.29}$$

i.e. in the symmetric, radiation (or Coulomb), respectively Lorentz gauges. It can easily be verified, using the Maxwell equations (2.1) that each of these potentials satisfies (3.1) for the corresponding e.m. field. Furthermore, as π is linear, we shall then find (πF) for the whole field F by simple addition of the various parts in (3.23) i.e.

$$(\pi F)(x) = A^{(o)}(x) + A^{(r)}(x) + A^{(j)}(x) \qquad .$$

We may now use this particular choice to calculate the compensating gauges defined by (3.7) and (3.8) and then, by (3.13), the representative factor set f^* defining the extension (3.17).
Using further the transformation law for the fields (2.2) and for the potentials (3.2), we get, after a long but straightforward calculation, that for the three parts of the field separately

$$\begin{aligned}
\chi_g(\pi(gF^{(o)}),x) &= (g(\pi F^{(o)}))_\mu(x)\, a^\mu + c_g - \tfrac{1}{2}(gF^{(o)})_{\sigma\rho} a^\sigma x^\rho + c_g \\
\chi_g(\pi(gF^{(r)}),x) &= \int^{x^o} g(\pi F^{(r)}))_o(x)\, dx^o + c'_g \qquad\qquad (3.30) \\
\chi_g(\pi(gF^{(j)}),x) &= c''_g
\end{aligned}$$

where $g = (a,\Lambda) \in IO(3,1)$ as before and where c_g, c'_g, c''_g are free integration constants, which may depend on πF but may obviously be chosen to be identically zero (as giving rise to equivalent extensions (3.17)). Further we obtain, with (3.13) and (3.30)

$$\begin{aligned}
(f^*(g_1,g_2))(\pi(gF^{(o)})) &= \tfrac{1}{2}(gF^{(o)})_{\sigma\rho}(\Lambda_1 a_2)^\sigma (a_1)^\rho \\
(f^*(g_1,g_2))(\pi(gF^{(r)})) &= 0 \qquad\qquad (3.31) \\
(f^*(g_1,g_2))(\pi(gF^{(j)})) &= 0
\end{aligned}$$

Using finally the linearity of the map π and of the Poincaré transformations, the compensating gauges and the factor set of the extension (3.17) are obtained by simple addition of the different parts of (3.30) and (3.31) so that finally

$$\begin{aligned}
\chi_g(\pi(gF),x) &= -\tfrac{1}{2}\,(gF^{(o)})_{\sigma\rho}a^\sigma x^\rho + \int^{x^o} (g(\pi F^{(r)}))_o(x)\,dx^o \\
(f^*(g_1,g_2))(\pi(gF)) &= \tfrac{1}{2}\,(gF^{(o)})_{\sigma\rho}(\Lambda_1 a_2)^\sigma (a_1)^\rho \qquad .
\end{aligned} \tag{3.32}$$

This result solves incidentally the problem of the possible factor systems involved by symmetry groups of e.m. fields [9]: indeed $(f^*(g_1,g_2))(\pi(gF))$ becomes then a constant $f(g_1,g_2)$, (F remaining of course unchanged under transformations of the symmetry group) and χ_g is then also constant as a function of πF. We get directly from (3.32), for all $g\in G_F$(see (2.3))

$$\begin{aligned}
\chi_g(x) &= -\tfrac{1}{2}\,F^{(o)}_{\sigma\rho}a^\sigma x^\rho + \int^{x^o} (g(\pi F^{(r)}))_o(x)\,dx^o \\
f(g_1,g_2) &= \tfrac{1}{2}\,F^{(o)}_{\sigma\rho}(\Lambda_1 a_2)^\sigma (a_1)^\rho \qquad .
\end{aligned} \tag{3.33}$$

These results look quite simple, seen the large class of fields we have considered, but are in fact quite important: they show in particular that the extension (3.17) is trivial unless the field has a Fourier component in the origin of the dual space whose contribution to the field is different of zero, i.e. for the class we consider, unless the field carries a constant uniform part. As a consequence, the relevant covariance operator group will as in the free case be homomorphic to the covering group of the Poincaré group, when and only when this contribution vanishes, and this fact is at our opinion quite remarkable.

4. The general covariance group $\mathbb{M}$.

In the previous section we have in a first step determined a covariance operator group by the analysis of its action on the potential subspace defined by a fixed cross-section π for an arbitrary but fixed external field and we have investigated some of its essential general structure properties. The group obtained in this way may however implicitely depend on the field we started with, as the function space Φ_F is generated by the factor system f^* corresponding to this given field. We have however in the sequel obtained an explicit expression for the corresponding factor system (see (3.32)) for an (almost) arbitrary external field so that we may now get rid of the above dependence, letting now F run over all possible fields and considering the most general function space generated by f^*. This function space will be denoted by $\mathbb{B}$. In this way each possible function in $\mathbb{B}$ will become an element of a new general covariance group $\mathbb{M}$. The explicit field dependence will then appear as we shall see in the value of that function at a point and will thus reappear at the level of the representations.

We observe for that purpose that, as follows from our result (3.32), only linear functions in the constant uniform part of the field do occur, so that this function space can be identified with the 6-dimensional dual vector space $T \wedge T$ of 4×4 antisymmetric contravariant real tensors, and is isomorphic to the original space of constant uniform fields. A basis for this functions space can thus be given by the (antisymmetric) external tensor product of a basis of the Minkovski space with itself, i.e.

$$E_{\mu\nu} = e_\mu \wedge e_\nu \tag{4.1}$$

with $\mu,\nu = 0,1,2,3$, and $(e_\mu)^\nu = \delta^\nu_\mu$. An arbitrary element B of $\mathbb{B}$ can then be expressed as

$$B = \sum_{\mu,\nu} B^{\mu\nu} E_{\mu\nu} \quad , \quad B^{\mu\nu} \in \mathbb{R} \quad , \quad \forall \ \mu,\nu \quad . \tag{4.2}$$

The group structure of $\mathbb{B}$ being generated because of (3.14) by the (additive) product of the real line, we have

$$(B_1 + B_2)^{\mu\nu} = B_1^{\mu\nu} + B_2^{\mu\nu} \quad .$$

An element of the general covariance group $\mathbb{M}$ can thus be written as a pair $\langle B,g \rangle$, $B \in \mathbb{B}$, $g \in IO(3,1)$ with product

$$\langle B,g \rangle \langle B',g' \rangle = \langle B + \zeta(g)B' + A(g,g'), gg' \rangle \tag{4.3}$$

where, as follows from (3.16), B' transforms under g as a contravariant tensor, i.e.

$$(\zeta(g)B')^{\sigma\rho} = (\Lambda^{-1})^{\sigma}_{\lambda} (\Lambda^{-1})^{\rho}_{\mu} (B')^{\lambda\mu} \tag{4.4}$$

and the general factor set $A(g,g')$ can be obtained from (3.15) and (3.32) as given by

$$A(g,g')^{\sigma\rho} = \tfrac{1}{2}\left[(\Lambda a')^{\sigma} \wedge a^{\rho}\right] = \tfrac{1}{4}\left[(\Lambda a')^{\sigma} a^{\rho} - (\Lambda a')^{\rho} a^{\sigma}\right] \quad . \tag{4.5}$$

In other words, $\mathbb{M}$ appears as an extension of $\mathbb{B} \cong \mathbb{R}^6$ by IO(3,1), i.e. the following sequence of groups is exact

$$0 \longrightarrow \mathbb{B} \longrightarrow \mathbb{M} \longrightarrow IO(3,1) \longrightarrow 1, \quad A,\zeta \quad . \tag{4.6}$$

The relation between $Q^*_{\pi F}$ and $\mathbb{M}$, for an arbitrary e.m. field F can then also be seen, using the preceding definitions and results, to be as illustrated in the following commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 \dashrightarrow & \mathbb{B} & \dashrightarrow & \mathbb{M} & \dashrightarrow & IO(3,1) & \rightarrow 1, & A\ ,\zeta & \\
& \Big\downarrow \rho_F & & \Big\downarrow & & \Big\| & & & (4.7) \\
0 \dashrightarrow & \Phi_{\pi F} & \dashrightarrow & Q^*_{\pi F} & \dashrightarrow & IO(3,1) & \rightarrow 1, & f^*,\zeta &
\end{array}
$$

where ρ_F is an epimorphism whose kernel is given by the elements of $\mathbb{B}$ which as functions of πF, i.e. as mappings

$$b : \pi(O_F) \dashrightarrow \mathcal{R}$$

where $b(\pi F) \equiv B \cdot F^{(o)} = B^{\sigma\rho} F^{(o)}_{\sigma\rho} \in \mathcal{R}$, vanish identically on O_F.

Similarly one may obtain directly the action of $\mathbb{M}$ on the potential subspace defined by π for a given F by the following homomorphism Σ (compare with (3.19)):

$$
\begin{aligned}
(\Sigma \langle B,g\rangle)(\pi F) &\overset{\text{def}}{=} \{\chi_g + b,g\}(\pi F) \\
&= g(\pi F) - \partial\left[\chi_g(\pi(gF),x) + b(\pi(gF))\right] \qquad (4.8) \\
&= \pi(gF)
\end{aligned}
$$

where b is as given in (4.8). In other words the covariance operator groups $Q^*_{\pi F}$ do appear, for each field, as representations operator groups of the general covariance group $\mathbb{M}$ as acting on the potential subspace defined by π.

We are of course now interested in the <u>physical representations</u> of $\mathbb{M}$ which we want to define explicitly. Assuming therefore that a change of gauge as acting on a physical state can only give a phase displacement, and assuming that, as in the free particle case, a Poincaré transformation acts on the x-variable via the substitution operator P_g, we define a mapping

$$V \; : \; M \text{ -- ---} \longrightarrow U(\mathcal{H})$$

from M to the unitary/antiunitary operators of some separable Hilbert space $\mathcal{H}$ (to be specified later on) of (here scalar) functions $\psi(x,\pi F)$ depending on x and (via the map π) on the field F. The mapping V is explicitly given (for $g \in P^{\dagger}$) by

$$V(\langle B,g\rangle)\psi(x,\pi F(x)) \stackrel{\text{def}}{=} \exp\{- i \frac{e}{c\hbar} (B\cdot gF^{(o)} + \chi_g(\pi(gF),x)\} \cdot$$

$$\cdot\ \psi(g^{-1}x,\pi F(g^{-1}x)) \qquad (4.9)$$

with $\chi_g(\pi(gF),x)$ as found in (3.32). If the functions $\psi(x,\pi F)$ describe the states of a charged particle in the external field F, and thus obey some equation of motion, represented by a linear differential operator O on $\mathcal{H}$, we have

$$O(x,\pi F)\ \psi(x,\pi F) = 0 \quad , \qquad \forall\ x \in M(4) \quad . \qquad (4.10)$$

We know, by construction, that the transformation $V(\langle B,g\rangle)$ must be a covariant transformation for this equation, corresponding to a change of reference frame. We have thus following the argument of section 1, and by definition of our group, that O will be a covariant equation of motion if and only if it satisfies the condition

$$V(\langle B,g\rangle)\ O(x,\pi F)\ (V(\langle B,g\rangle))^{-1} = O(g^{-1}x,\pi' F) =$$

$$= O'(x,\pi F) \equiv O(x,\pi gF) \quad . \qquad (4.11)$$

The equation (1.1)' tells us now that if $\psi(x,\pi F)$ is a solution of (4.10),

$$V(\langle B,g\rangle)\ \psi(x,\pi F) \equiv \Phi(x,\pi gF)$$

is then a solution in the new frame thus satisfies the equation

$$O(x,\pi gF)\ \Phi(x,\pi gF) = 0 \quad . \tag{4.12}$$

Using these relations and the fact that, by (3.13), (3.32), and (4.5)

$$A(g,g')\cdot gg'F^{(o)} = f^{*}(g,g')(\pi(gg'F)) \equiv \chi_{g}(\pi(gg'F),x) +$$

$$+ \psi(g)\chi_{g'}(\pi(g'F),x) - \chi_{gg'}(\pi((gg')F),x) \tag{4.13}$$

one may verify, after a short calculation, that

$$V(\langle B,g\rangle\langle B',g'\rangle)\psi(x,\pi F) = V(\langle B,g\rangle)V(\langle B',g'\rangle)\psi(x,\pi F) \tag{4.14}$$

so that V is indeed an homomorphism and $\mathcal{H}$ is a representation space for the general covariance group M. In analogy with the free particle case, the quantum system described by this space will be called an elementary particle in interaction with the field F, if it is irreducible.

As a general result, we have thus found that in the presence of an external e.m. field, the relevant covariance group is no longer the Poincaré group but contains the Poincaré groups only as a factor group. Since our approach is in fact independent of any specific equation of motion it will be interesting to analyze more in detail the structure of this group M and of its representation. It will be possible in this way not only to obtain information on the covariant equations of motion in the presence of an external field but to extend usefully the group theoretical definition of an elementary particle for this case where, as in a new world, an external field is present. This means, in other words, that we will be in position to analyse what are also the more conceptual consequences of the external approximation and how covariant equations of motion can be characterized. This will be done in a subsequent paper [2]. Anticipating a little, let us however mention already here that, if the Klein-Gordon and Dirac equations with minimal coupling are in fact covariant under M, this will (surprisingly perhaps) not necessarily be so for the usual higher spin equations ($s \geqslant 1$), minimally coupled to a constant uniform magnetic field for example. As we shall then see, this fact can also be related to the so-called a-causality troubles characterizing these equations [3,4].

References.

[1] E.P. Wigner, Ann. of Math. 40, 149 (39)

[2] N. Giovannini, Chapter V of the present treatise

[3] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (69) and 188, 2218 (69)

[4] "Troubles in the external field problem for invariant wave equations", Tracts in Math. and Nat. Sc. Vol. 4, rev. A.S. Wightman, Gordon and Breach N.Y. (1971)

[5] U. Cattaneo and A. Janner, J. Math. Phys. 15, 1155 and 1166 (74)

[6] G. Rideau, Phys. Scripta 9, 163 (74)

[7] see e.g. ref. [4], p. 21

[8] A. Janner and T. Janssen, Physica 60, 292 (72)

[9] A. Janner and T. Janssen, Physica 53, 1 (71)

[10] M. Boon, J. Math. Phys. 13, 1268 (72)

[11] N. Giovannini, Proc. 3rd Int. Coll. on Group Theoretical Methods in Physics, Marseille (1974), p.336

[12] N. Giovannini, Chapter I of the present treatise.

CHAPTER V

ELEMENTARY PARTICLES AS REPRESENTATIONS OF THE COVARIANCE GROUP IN THE PRESENCE OF AN EXTERNAL ELECTROMAGNETIC FIELD

Introduction.

In the previous chapter, we have derived, independently of any equation of motion, a general covariance group for a charged relativistic particle moving in an (almost) arbitrary external electromagnetic field, essentially on the simple basis of the invariance of the Maxwell equations under Poincaré transformations. Let us briefly remind here how this group was constructed. The usual procedure, where one considers the problem of a charged particle moving in such a field is to modify the free equation of motion (Klein-Gordon, Dirac etc.) by so-called minimal coupling, introducing thus a potential. Since potentials are not observables as fields are, there is in fact a certain arbitrariness in the equation of motion, as for example the transformation law of a potential under a Poincaré group element is not unique and as there does not correspond one but an infinite set of potentials to a given field. As a consequence Poincaré covariance has only to hold for an infinite class of (physically indistinguishable) equations of motion and not for each element of that class separately. This implies incidentally also that arbitrary gauge transformations are then covariant transformations (for this class) too, this giving rise, as is well known, to interesting but difficult to handle infinite dimensional groups. One could on the other side try to avoid potentials and construct a pure field dependent formalism as it has been shown to be possible for example in the Dirac case [2]. In [1] we had chosen a way inbetween, using potentials, but getting rid of the arbitrariness mentioned above by defining group elements which, as acting on a potential, did not change the (arbitrary but fixed) gauge of the chosen potential and consisted then of coupled gauge and Poincaré transformations. The resulting group was shown to be independent of the reference frame and of the gauge chosen and also to contain the Poincaré group only as a (non-trivial) factor group (i.e. not as a subgroup).

In this chapter we deduce systematically all projective unitary/antiunitary (continuous) irreducible representations (short PUAIR) of this group and discuss their physical interpretation and some of the physical consequencesof the results. In particular we show how our approach leads to a possible solution of the so-called a-causality troubles, for particles of spin equal or larger than 1 [3-4]. The Klein-Gordon and Dirac equations, however, minimally coupled to the external field, are shown to correspond to representations of this group, i.e. to transform covariantly.

This chapter will be organized as follows: in the first part we briefly remind the structure of our general covariance group and analyse it in some more detail. In part two we derive all irreducible unitary representations of a nilpotent normal subgroup of this group and, in part three, we induce these representations to the connected component to one of the whole group. In part four we include the so-called discrete transformations (related to space and time-reversals), too, and finally in part five we discuss some physical aspects and consequences of our results.

1. The structure of the general covariance group $\mathbb{M}$.

Before we determine the PUAIR of $\mathbb{M}$, let us first remind its definition, as found in [1], and indicate in some more detail its structure. We write the elements $m \in \mathbb{M}$ in the form

$$m = \langle B, a, \Lambda\rangle \qquad (1.1)$$

with $B \in T \wedge T$, the (antisymmetric) external Kronecker product of the Minkovski space $M(4)$ with itself ($B = B^{\mu\nu}E_{\mu\nu}$, $B^{\mu\nu} \in \mathbb{R}$, $\forall\ \mu,\nu = 0,1,2,3$ $E_{\mu\nu} = e_\mu \wedge e_\nu$, $\{e_\mu\}$ a basis of $M(4)$, $(e_\mu)^\nu = \delta^\nu_\mu$ and $(a,\Lambda) \equiv g \in IO(3,1)$, an element of the Poincaré group, $a \in U$, a space-time-translation, $\Lambda \in O(3,1)$, an element of the homogeneous Lorentz group [1)]. The product in $\mathbb{M}$ is then given by

$$m \cdot m' = \langle B + \zeta(g)B' + A(g,g'), a + \Lambda a', \Lambda\Lambda'\rangle \qquad (1.2)$$

---

1) For elements of the Poincaré group $IO(3,1)$, we use the same conventions as in [1], i.e. $(a,\Lambda)x = \Lambda x + a$ for x a 4-vector, with then $(a,\Lambda)(a',\Lambda') = (a + \Lambda a',\Lambda\Lambda')$; further $(\Lambda x)_\nu = \Lambda^\mu_{\ \nu}x_\mu$ for covariant, and $(\Lambda x)^\mu = \Lambda^{-1\mu}_{\ \ \ \nu}x^\nu$ for contravariant vector components.

with $(\zeta(g)B')^{\mu\nu} = \Lambda^{-1\mu}_{\ \ \rho}\,\Lambda^{-1\nu}_{\ \ \sigma}\,(B')^{\rho\sigma}$ and $A(g,g')$ a factor system, defined by

$$A(g,g')^{\sigma\rho} = \tfrac{1}{2}((\Lambda a')^{\sigma} \wedge a^{\rho}) = \tfrac{1}{4}((\Lambda a')^{\sigma}a^{\rho} - (\Lambda a')^{\rho}a^{\sigma}). \quad (1.3)$$

The inverse of (1.1) is then easily found from (1.2) and (1.3) to be equal to

$$m^{-1} = \langle -\zeta(g^{-1})B,\ -\Lambda^{-1}a,\ \Lambda^{-1}\rangle \quad (1.4)$$

because it follows from (1.3) that $A(g,g^{-1}) = 0$.

We remind also here that the physical representations of $\mathbb{M}$ were defined on a (separable) Hilbert space of (here scalar) functions ψ of x and F(x) (the external field) by (see [1] sections 1 and 4)

$$V(\langle B,a,\Lambda\rangle)\psi(x,F(x)) \equiv \Phi(x,(gF)(x)) =$$

$$= \exp\{-i\frac{e}{c\hbar}(B\cdot gF^{(o)} + \chi_g(\pi(gF),x))\}\psi(g^{-1}x,F(g^{-1}x)) \quad (1.5)$$

with $B\cdot gF^{(o)} \equiv B^{\sigma\rho}(gF^{(o)})_{\sigma\rho} \in \mathbb{R}$, $\chi_g(\pi(gF),x)$ the compensating gauge and $F^{(o)}$ the c.u. part of the field as defined in [1]. For ψ a solution of a covariant equation of motion in the field F, Φ defined by (1.5) was shown to be a solution in the field gF. The generalization to more components wave functions is then similar as in [1] (see also section 5 of the present Chapter).

Denoting now by $\mathbb{B}$ the subgroup of $\mathbb{M}$ consisting of all elements of the form $\langle B,0,1\rangle$ ($\mathbb{B} \cong \mathbb{R}^6$) and by K the subgroup of $\mathbb{M}$ generated by all $\langle B,a,1\rangle$, $a \in U$ the following relations are easily verified:

(i) $\mathbb{B} \lhd K$, ($\mathbb{B}$ normal in K) and $K/\mathbb{B} \cong U$, i.e. K appears as an extension of $\mathbb{B}$ by U; this extension is characterized by a factor set A(a,a'), given by the restriction to U x U of (1.3).

(ii) $\mathbb{B} \lhd \mathbb{M}$, $\mathbb{M}/\mathbb{B} \cong IO(3,1)$, with as corresponding factor set A(g,g') as in (1.3).

(iii) $K \lhd \mathbb{M}$, $\mathbb{M}/K \cong O(3,1)$, giving rise to a semidirect product

(iv) $U \lhd IO(3,1)$, $IO(3,1)/U \cong O(3,1)$, corresponding also to a semidirect product.

These properties are resumed in the following diagram of exact sequences.

$$\begin{tikzcd}
 & & U \arrow[dr, tail] & & \\
 & K \arrow[ur, two heads] \arrow[dr, tail] & & IO(3,1) \arrow[ddr, two heads] & \\
 & & \mathbb{M} \arrow[ur, two heads] \arrow[drr, two heads] & & \\
\mathbb{B} \arrow[uur, tail] \arrow[urr, tail] & & & & O(3,1)
\end{tikzcd} \qquad (1.6)$$

$\rightarrowtail$ denotes a monomorphism and $\twoheadrightarrow$ an epimorphism

We note further that, as can also easily be seen

(v) $\mathbf{B}$ and U are abelian

(vi) K is nilpotent, with a lower central series of length 2.

We are, because a physical state will be described by a ray in a separable Hilbert space $\mathcal{H}$ [1], not interested in the proper representations of $\mathbf{M}$ but in the projective (up to a factor) ones. These can be obtained, since $\mathbf{M}$ is separable and locally compact, by considering the ordinary representations of a larger group $\mathbf{M}^{\sigma}$, by the procedure of lifting [5,6], where σ is a multiplier over $\mathbf{M}$ with respect to the UA decomposition $\mathbf{M}^{\uparrow} \cup \mathbf{M}^{\downarrow}$ ($\mathbf{M}^{\uparrow}$ being the subgroup of $\mathbf{M}$ corresponding to orthochronous Poincaré transformations only). This multiplier satisfies thus the relations

$$\sigma(m_1,m_2m_3)\ \sigma^{m_1}(m_2,m_3) = \sigma(m_1,m_2)\sigma(m_1m_2,m_3) \quad , \qquad \forall\, m_1,\, m_2,\, m_3 \in \mathbf{M} \tag{1.7}$$

with

$$\sigma^{m_1} = \begin{cases} \bar{\sigma}\ , & m_1 \in \mathbf{M}^{\downarrow} \\ \sigma\ , & m_1 \in \mathbf{M}^{\uparrow} \end{cases}$$ ($\bar{\sigma}$ the complex conjugate of σ).

Such a multiplier is sometimes also called a co-multiplier; we keep however the same nomenclature as in [6]. Because of the special role played by discrete elements, and in particular because of the (assumed) anti-unitary character of transformations containing time-inversion we consider in a first step the connected component $\mathbf{M}_o^{\sigma}$ of $\mathbf{M}^{\sigma}$ only (the remaining discrete transformations will be reintroduced later on) The group $\mathbf{M}_o^{\sigma}$ depends of course on σ and it is more useful, whenever possible, to consider a larger group $\mathbf{M}_o^{\Sigma}$ which is an extension of the multiplier group Σ (the group of all inequivalent possible σ) by M_0, the connected component to one of M), and whose ordinary representations describe all projective inequivalent representations of M_0. It follows from theorems 3.2 and 4.1 of Bargmann [7] and from the fact that the second cohomology group $H^2(\underline{M}, \mathbb{R}) = 0$ 1), where $\underline{M}$ is the Lie-algebra of M_0, that all projective inequivalent representations of M_0 can be obtained from the ordinary ones of the covering group $\overline{M}_0$ of M_0, which has the following structure: it has elements $\langle B, a, \overline{\Lambda}\rangle$ with $\overline{\Lambda} \in SL(2,\mathbb{C})$, B and a as before, K being its own covering group. In an analogous way as in (1.6) we can illustrate the structure of $\overline{M}_0$ with the following diagram of exact sequences:

$$\begin{tikzcd}
 & & U \arrow[dr] & & \\
 & K \arrow[ur] \arrow[dr] & & \overline{IO(3,1)} \arrow[ddr] & \\
 & & \overline{M}_0 \arrow[ur] \arrow[drr] & & \\
IB \arrow[uur] \arrow[urr] & & & & SL(2,C)
\end{tikzcd} \qquad (1.8)$$

As we can see from this diagram, a possible way for obtaining all irreducible unitary representations of $\overline{M}_0$ will be by inducing, in the sense of Mackey [5], from the ones of K. The latter ones can be obtained following the theory of Kirillov [9,10], since K is nilpotent.

---

1) We omit here the proof since this result was already obtained by Schrader [8]. Since other results of this paper overlap some of the results to follow, we shall discuss it more in detail later on.

Let us now determine the Lie-algebra of $\mathbb{M}$. We use the following notation for the generators: let $M_{\mu\nu}$ generate a rotation in the μ-ν plane of space-time ($M_{\mu\nu} \in s\ell(2,\mathbb{C})$), Π_μ a translation and, for $\mathbb{B}$, let $\mathbb{F}_{\mu\nu}$ be the infinitesimal generator of the group element $\Xi_{\mu\nu}$. The Lie-algebra is then easily obtained, using the results concerning the group as obtained so far, as given by

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} - g_{\nu\rho} M_{\mu\sigma} - g_{\mu\sigma} M_{\nu\rho} \\
[M_{\mu\nu}, \mathbb{F}_{\rho\sigma}] &= g_{\mu\rho} \mathbb{F}_{\nu\sigma} + g_{\nu\sigma} \mathbb{F}_{\mu\rho} - g_{\nu\rho} \mathbb{F}_{\mu\sigma} - g_{\mu\sigma} \mathbb{F}_{\nu\rho} \\
[\Pi_\mu, \Pi_\nu] &= \mathbb{F}_{\mu\nu} \qquad\qquad (1.9) \\
[M_{\mu\nu}, \Pi_\sigma] &= g_{\mu\sigma} \Pi_\nu - g_{\nu\sigma} \Pi_\mu
\end{aligned}
$$

where the metric $g_{\mu\nu}$ is given by $g_{oo} = - g_{ii} = -1$, $i = 1,2,3$. All other commutators vanish.

This 16-dimensional Lie-algebra is actually not unknown and has even a (very adapted) name, as proposed first by S.L. Glashow (see Stein [11]): the Maxwell Lie-algebra. It has been quoted by Bacry et al. [12] and studied in some more detail by Schrader [8] in an actually quite different, or better said, much more specific, context. First, all these authors consider only constant uniform (c.u.) e.m. fields whereas we are dealing with a very large class of (also inhomogeneous) fields (see [1]). Second the generators $\mathbb{F}_{\mu\nu}$ are identified by them with the eigenvalues they take in the presence of a given c.u. field (namely with the field components, as we shall see later on), and for representations generated from a given equation of motion (Klein-Gordon or Dirac with minimal coupling). This indicates incidentally, since we have made no use of such an equation, that these equations will characterize representations of our group, i.e. are covariant under $\mathbb{M}$, as we shall see more in detail in section 5.

Our derivation and interpretation are clearly different and our goals somehow more ambitious. We shall discuss later on some interesting remarks made in [12]. More interesting at the moment are the results of Schrader: in his paper the irreducible unitary representations of the subgroup K of M are derived. Unfortunately, only the case of a c.u. magnetic field (and its Poincaré transforms) is then considered and only in two representations (as obtained in fact from the Klein-Gordon and Dirac equations).

Because we want to derive all PUAIR of M, we shall, for completeness, calculate them from the beginning, including (in a somehow different and simpler way) those results obtained by Schrader concerning K that we shall need in the sequel, too.

2. The irreducible representations of the subgroup K of M.

These representations can be completely determined using the theory of Kirillov [9,10] for connected (here simply connected) nilpotent Lie groups. Let us therefore first briefly remind the general procedure.

Let $\underline{k}$ be the Lie algebra of K, $\underline{k}'$ the dual space of $\underline{k}$ and $\mathrm{coAd}_{\underline{k}'}(K)$ the coadjoint representation of K on $\underline{k}'$ (the contragredient of the adjoint representation), which for $\omega \in \underline{k}'$ (i.e. for a linear form on $\underline{k}$) is defined by

$$(\mathrm{coAd}_{\underline{k}'}(k)\omega)(\xi) \overset{\mathrm{def}}{=} \omega(\mathrm{Ad}_{\underline{k}}(k)^{-1}\xi) \qquad (2.1)$$

with $k \in K$, $\xi \in \underline{k}$. Let then O_ω denote the orbit of an element $\omega \in \underline{k}'$, i.e. the set of all images of ω under the action of K as defined by (2.1). Since (2.1) is a representation of K, the set $\{O_\omega\}$ is a partition of $\underline{k}'$. Consider now in each orbit one arbitrary (but fixed) element ω and consider a subalgebra $\underline{\ell} \subseteq \underline{k}$ satisfying

$$[\underline{\ell}\ ,\ \underline{\ell}] \subsetneq \text{Ker}\ \omega \tag{2.2}$$

such a subalgebra $\underline{\ell}$ is called <u>subordinate to ω.</u> The following map T_ω on the Lie-subgroup L of K generated by $\underline{\ell}$ ($L = \exp \underline{\ell}$)

$$T_\omega(\exp x) \stackrel{\text{def}}{=} \exp i\ \omega(x) \quad , \qquad x \in \underline{\ell} \tag{2.3}$$

is then clearly a one dimensional unitary representation of L. This representation can then be induced, in the canonical way, to a representation $V_{\omega,\underline{\ell}}$ of K by

$$\begin{aligned} (V_{\omega,\underline{\ell}})(k)\ \varphi(\lambda) &\equiv (T_\omega \uparrow K)(k)\ \varphi(\lambda) \\ &= T_\omega(\lambda\ k(\lambda')^{-1})\ \varphi(\lambda') \end{aligned} \tag{2.4}$$

where λ, λ' are (fixed) representatives of the (right) coset decomposition K/L, with λ' determined by the condition $\lambda k(\lambda')^{-1} \in L$, and $\varphi(\lambda)$ is a measurable, quadratic integrable function on this coset space, with respect to some (quasi-)invariant (under the action of K) ergodic measure μ, and with values in the carrier space of T_ω. This measure is then unique, as a class. Since the coset space and a set of coset representatives are in one-to-one Borel correspondence with each other we use, or better said we abuse, as is usual, the parameter λ to describe both spaces.

The following then holds

Theorem 2.1. (Kirillov [10])

(i) $V_{\omega,\underline{\ell}}$ is irreducible if and only if dim $\underline{\ell}$ is maximal

(ii) any irreducible representation of K can be obtained in this way, up to equivalence

(iii) $V_{\omega,\underline{\ell}}$ and $V_{\omega',\underline{\ell}'}$, both irreducible are equivalent, if and only if $O_\omega = O_{\omega'}$

(iv) $\hat{K}$ the dual of K is always of Type I .

A subalgebra $\underline{\ell}$ subordinate to a linear form ω and of maximal dimension is also called a real polarisation at ω. From now on only such subordinate subalgebras will be considered.

In the sequel, the following result will also be helpful:

Theorem 2.2. [13]. Whenever $\underline{\ell}$, real polarisation at ω, can be chosen ideal, then the following holds:

(i) $\underline{\ell}$ is his own centralisator with respect to ω, i.e. if $\xi \in \underline{k}$,

$$\omega([\underline{\ell},\xi]) = 0 \quad <=> \quad \xi \in \underline{\ell}$$

(ii) the coset space K/L is Borel isomorphic with the orbit of T_ω under K and is 1-to-1 characterized by the classes $\bar{\omega}$ of elements ω in O_ω which coincide with each other when restricted to $\underline{\ell}$.

Let us now apply this theory to our group K: an element $\omega \in \underline{k}'$ can be characterized by an antisymmetric tensor $f_{\mu\nu}$ and a vector p_μ ($\mu,\nu = 0,1,2,3$), and will be denoted by (f,p). Its action on $\underline{k}$ is defined by

$$(f,p)(B^{\mu\nu}F_{\mu\nu} + a^\mu P_\mu) \stackrel{def}{=} B^{\mu\nu}f_{\mu\nu} + a^\mu p_\mu, \quad \mu,\nu = 0,1,2,3. \quad (2.5)$$

Using the commutation relations (1.9) one obtains after a short calculation that the adjoint representation is given, for $k = \langle B,a,1\rangle \in K$, by

$$\mathrm{Ad}_{\underline{k}}(\langle B,a,1\rangle)((B')^{\rho\sigma}\mathbb{F}_{\rho\sigma} + a'^{\sigma}\Pi_{\sigma}) = (B'^{\mu\nu} + (a \wedge a')^{\mu\nu})\mathbb{F}_{\mu\nu} + a'^{\mu}\Pi_{\mu} \tag{2.6}$$

so that, by (2.1) and (2.5)

$$\mathrm{coAd}_{\underline{k}'}(\langle B,a,1\rangle)(f,p) = (f,p + a\cdot f) \tag{2.7}$$

with $(a\cdot f)_{\mu} = a^{\nu}f_{\nu\mu} \overset{\mathrm{def}}{=} (f(a))_{\mu}$, f being in this last expression considered as a linear map from U to $M^*(4)$, the dual Minkovski space. The orbits are thus given by $O_{(f,p)} = \{(f,p + a\cdot f) \mid \forall\, a \in U\}$ and are characterized by a tensor f and a manifold of vectors p modulo Im(f). The explicit form of Imf depends of course on f and needs to be investigated in more details, as we shall do now: because f is antisymmetric, it has, as a bilinear form, necessarily an image of even dimension. We may thus distinguish the following three cases, in turn:

(α) dim (Im(f)) = 0, then necessarily f = 0 and the corresponding orbits are completely characterized by a 4-vector $p \in M^*(4)$ and will therefore be denoted O_p. This case will correspond of course also to free particles and will give rise to the well known PUA representations of the Poincaré group. It will thus only be shortly mentioned for completeness in the sequel.

(β) dim (Im(f)) = 4, then necessarily det f $\neq$ 0. As a consequence, for any p, there exists an $a \in U$ such that $p = -\,a\cdot f$ so that each such orbit goes through each point in $M^*(4)$ and the orbit is thus completely characterized by the tensor f (with det f $\neq$ 0). These orbits are denoted O_f.

(γ) dim (Im(f)) = 2, then det f = 0, f $\neq$ 0, and there exists, because dim (Ker(f)) = 4-dim (Im(f)) = 2, a 2-dimensional subspace of U generated by two linear independent (but of course not unique) translations $a^{(1)}$ and $a^{(2)}$ such that $\forall\ \mu_1,\mu_2 \in \mathbb{R}$

$$a(\mu_1,\mu_2)\cdot f \;\equiv\; (\mu_1 a^{(1)} + \mu_2 a^{(2)})\cdot f \;=\; 0 \quad . \tag{2.8}$$

These orbits are thus completely characterized by an antisymmetric tensor f with det f = 0, f $\neq$ 0 and an element q of the 2-dimensional factor space $M^*(4)/\mathrm{Im}(f)$. They will be denoted $C_{f,q}$.

Let us now calculate subalgebras $\underline{\ell}_\omega$ subordinate to $\omega \in \underline{k}'$ and of maximal dimensions with ω an (arbitrary but fixed) representant of each of these orbits. We first have, because of (2.2), to calculate the commutator algebra of $\underline{k}$. It follows readily from (1.9) that

$$[\underline{k}\ ,\ \underline{k}] \;=\; \{a^\mu a'^\nu F_{\mu\nu},\ \forall\, a,a' \in U\} \quad . \tag{2.9}$$

Considering again the different cases in turn, we find then:

(α) f = 0 so that trivially $\underline{\ell}_{(o,p)} = \underline{k}$, using (2.9)

(β) Because det f $\neq$ 0, there exist two, and no more (see [8,14]), vectors $a^{(1)}$ and $a^{(2)}$ such that $(a^{(1)})^\mu (a^{(2)})^\nu f_{\mu\nu} = 0$, with $a^{(1)}$ arbitrary and $a^{(2)}$ determined by this condition. Denoting by $a(\lambda_1,\lambda_2)$ an element $\lambda_1 a^{(1)} + \lambda_2 a^{(2)}$ in the subspace generated by $a^{(1)}$ and $a^{(2)}$ we have then $\forall\ \lambda_i,\lambda_i' \in \mathbb{R}$, i = 1,2

$$a^\mu(\lambda_1,\lambda_2)\ a^\nu(\lambda_1',\lambda_2')\ f_{\mu\nu} \;=\; 0 \quad . \tag{2.10}$$

Choosing now as representative of each orbit O_f the element (f,0) we have Ker f = 0, thus Ker (f,0) = $\{a^\mu \Pi_\mu \,|\, \forall a \in U\}$ so that, using (2.9) and and (2.10), we obtain

$$\underline{\ell}_{(f,0)} = \{a^{\mu}(\lambda_1,\lambda_2)\Pi_{\mu} + B^{\mu\nu}\mathbb{F}_{\mu\nu} | \forall B^{\mu\nu} \text{ and } \lambda_1,\lambda_2 \in \mathbb{R}\} \qquad (2.11)$$

(γ) Because of (2.8) and because $f \neq 0$, it is always possible to choose one (and no more, see again [8, 14]) additional vector $a^{(3)}$ such that with $a^{(1)}$ and $a^{(2)}$ as in (2.8) we have $(a^{(i)})^{\mu}\,(a^{(j)})^{\nu}\,f_{\mu\nu} = 0, i,j \in 1,2,3$. Denoting by $a(\mu_1,\mu_2,\mu_3)$ an arbitrary element of the form $\mu_1 a^{(1)} + \mu_2 a^{(2)} + \mu_3 a^{(3)}$, $\mu_i \in \mathbb{R}$ we have then, $\forall \mu_i,\mu_i' \in \mathbb{R}$

$$a^{\mu}(\mu_1,\mu_2,\mu_3) \cdot a^{\nu}(\mu_1',\mu_2',\mu_3') \cdot f_{\mu\nu} = 0 \qquad (2.12)$$

so that, for any representant $0_{f,q}$ of each of these orbits, we obtain, using once more (2.9),

$$\underline{\ell}_{(f,p)} = \{a^{\mu}(\mu_1,\mu_2,\mu_3)\Pi_{\mu} + B^{\mu\nu}\mathbb{F}_{\mu\nu} | \forall B^{\mu\nu} \text{ and } \mu_i \in \mathbb{R}\}\,. \qquad (2.13)$$

The corresponding subgroups L_{ω} of K are thus given, for the three cases separately

$$(\alpha)\quad L_{(0,p)} = \{<B,a,1>\} = K$$

$$(\beta)\quad L_{(f,0)} = \{<B,a(\lambda_1,\lambda_2),1>\} \lhd K \qquad (2.14)$$

$$(\gamma)\quad L_{(f,p)} = \{<B,a(\mu_1,\mu_2,\mu_3),1>\} \lhd K$$

and are, by construction, of maximal dimension and normal in K.

The corresponding representations T_{ω} of these groups L_{ω} are then, using (2.3), given by

$$(\alpha)\quad T_{(0,p)}(<B,a,1>) = \exp i\,(0,p)(B^{\mu\nu}\mathbb{F}_{\mu\nu} + a^{\mu}\Pi_{\mu}) = \exp i\,(p.a) \qquad (2.15)$$

$$(\beta)\quad T_{(f,0)}(\langle B,a,1\rangle) = \exp i\,(f,0)(B^{\mu\nu}F_{\mu\nu} + a^{\mu}(\lambda_1,\lambda_2)\Gamma_{\mu})$$
$$= \exp i\,(B\cdot f) \qquad (2.15)$$

$$(\gamma)\quad T_{(f,p)}(\langle B,a,1\rangle) = \exp i\,(f,p)(B^{\mu\nu}F_{\mu\nu} + a^{\mu}(\mu_1,\mu_2,\mu_3)T_{\mu})$$
$$= \exp i\,(B\cdot f + a(\mu_1,\mu_2,\mu_3)\cdot p) \quad .$$

These representations can now be explicitly induced to **K** as described in (2.4). We again consider the different cases in turn

(α) $L_{(0,p)} = K$ so that the coset space consists of a single point and trivially

$$V_{(0,p)}(\langle B,a,1\rangle) = \exp i(p\cdot a) \qquad (2.16)$$

i.e. we get a <u>4-dimensional family of one-dimensional representations of **K**</u> parametrized by a vector $p \in M^*(4)$.

(β) The coset space K/L is two dimensional and can be identified with the translation subspace generated by two vectors $a^{(3)}$ and $a^{(4)}$, chosen so to complete $a^{(1)}$ and $a^{(2)}$ in (2.10) to a basis of U. We decompose thus any translation $a \in U$ as $a \equiv a(\lambda_1,\ldots,\lambda_4) = \sum_{i=1}^{4} \lambda_i a^{(i)}$ and we parametrize the coset space by the same λ_3 and λ_4. The function φ of (2.4) is then a quadratic integrable measurable function on $\mathbb{R}^2$ with respect to the invariant ergodic (here Lebesgue) measure μ. Denoting then an element $k \in \mathbf{K}$ by $\langle B,a(\lambda_1,\lambda_2,\lambda_3,\lambda_4),1\rangle$ the coset condition reads

$$\langle 0,a(\lambda_3',\lambda_4'),1\rangle \quad \langle B,a(\lambda_1,\ldots,\lambda_4),1\rangle \quad \langle 0,a(\lambda_3'',\lambda_4''),1\rangle^{-1} \in L_{(f,0)}$$

i.e., using the group product (1.2) and the result (2.14)

$$a(\lambda_3'',\lambda_4'') = a(\lambda_3,\lambda_4) + a(\lambda_3',\lambda_4') = a(\lambda_3+ \lambda_3', \lambda_4+ \lambda_4') \quad .$$

We then obtain for the induced representations, using (2.4)

$$V_{(f,0)}(\langle B,a(\lambda_1,\ldots,\lambda_4),1\rangle)\ \varphi(\lambda_3',\lambda_4') = \\ = T_{(f,0)}(\langle B + A(a_1,a_2,a_3),a_2,1\rangle)\ \varphi(\lambda_3+ \lambda_3',\lambda_4 + \lambda_4') \tag{2.17}$$

where $a_1 \equiv a(\lambda_3',\lambda_4')$, $a_2 \equiv a(\lambda_1,\lambda_2)$, $a_3 \equiv a(\lambda_3,\lambda_4)$. Furthermore $A(a_1,a_2,a_3)$ is given, using the group product rule and the linearity and antisymmetry properties of the factor set $A(a,a')$, the restriction to U x U of the factor set in (1.3), by

$$A(a_1,a_2,a_3) = A(a_1,a_2+ a_3) + A(a_1+ a_3,a_2) \tag{2.18}$$

With (2.17) we have now found a <u>6-dimensional family of infinite dimensional representations of K</u> parametrized by the 6-dimensional linear space of functions $f_{\mu\nu}$, where the hypersurface characterized by det $f = 0$ is left out.

(γ) The coset space K/L is 1-dimensional and can be identified, completely analogously as in the previous case, with the subspace generalated by a fourth translation $a^{(4)}$ completing $a^{(1)}$ $a^{(2)}$ $a^{(3)}$ of (2.12) to a basis. The functions φ are now (quadratic integrable and measurable) functions on $\mathbb{R}$ with respect to the invariant ergodic (Lebesgue) measure. We obtain similarly to (2.17), with now $a = a(\mu_1,\ldots,\mu_4) = = \sum_{i=1}^{4} \mu_i\ a^{(i)}$:

$$V_{(f,p)}(\langle B,a(\mu_1,\ldots,\mu_4),1\rangle)\ \varphi(\mu_4') \\ = \Gamma_{(f,p)}(\langle B + A(a_1,a_2,a_3),a_2,1\rangle)\ \varphi(\mu_4+ \mu_4') \tag{2.19}$$

where $A(a_1,a_2,a_3)$ is formally as in (2.18) with but now $a_1 = a(\mu_4')$, $a_2 = a(\mu_1,\mu_2,\mu_3)$, $a_3 = a(\mu_4)$.

In (2.19) we have found a <u>7-dimensional family of infinite dimensional representations of K</u>, characterized by the 5-dimensional hypersurface in the $f_{\mu\nu}$ space with det $f = 0$ (where the point $f = 0$ is obviously left out), and the 2-dimensional factor space $M^*(4)/\mathrm{Im}(f)$.

Before we induce these representations to $\bar{\mathbb{M}}_o$, let us note that the dual $\hat{K}$ of K is smooth, in the sense of Mackey [5]. This can be verified, using the facts that $K^c = \{V_{(f,p)} |$ one (f,p) per orbit$\}$ is Borel isomorphic to the 10-dimensional space $\underline{k}'$, which is complete, separable and isomorphic to a metric space thus standard (as a Borel space): the set of representations we have chosen is indeed a Borel cross-section and the result follows then from the theorem 8.5 of [5]. It has actually been shown, since, that, as conjectured by Mackey, this is equivalent to the fact that K is of Type I [15], and the last property follows from theorem 2.1. We therefore drop the details.

The fact that $\hat{K}$ is smooth and of Type I is however of importance as it will allow us to apply the general theory of Mackey [5] concerning the representations by induction of group extensions. This is what we shall do in the next section.

3. Induction to the connected part $\overline{M}_o$ of M

a) Representations of group extensions

The theory of Mackey for the induction of representations is a well known procedure, at least when applied to a (regular) semi-direct product $G = N \wedge_\varphi H$, with G separable locally compact and N abelian (φ denoting some (given) homomorphism from H to Aut N, the group of automorphisms of N) [16]. Less known perhaps is the more general case where G, separable locally compact, is any (regular) extension of a group N, not necessarily abelian, by a group H, i.e. appears in the following exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1, \quad m,\varphi \tag{3.1}$$

characterized by a factor set $m : H \times H \longrightarrow N$ with $m(h_1,h_2h_3) + \varphi(h_1)\, m(h_2,h_3) = m(h_1,h_2) + m(h_1h_2,h_3)$ and a map $\varphi : H \longrightarrow \text{Aut } N$ satisfying the condition $\varphi(h_1)\varphi(h_2) = \mu(m(h_1,h_2))\, \varphi(h_1h_2)$, μ being the canonical epimorphism from N to In N, the group of inner automorphisms of N. The essential difference, in this more general case is that, as shown by Mackey [5], whether N is not abelian or whether the extension (3.1) does not split, no longer ordinary representations of the adequate subgroups of H have to be considered but certain projective ones. Since we shall need in the sequel the explicit formulas of the general theory, and in particular an explicit expression for the factor sets involved, we first indicate briefly and in a way convenient for our purposes, the general construction procedure. As, however, in our problem m = 0, we shall restrict ourselves here to this more special case.

Let therefore $\hat{N}$ be the dual of N in (3.1), $[\hat{n}] \in \hat{N}$, $\hat{n}$ a representant of the corresponding class $[\hat{n}]$ of irreducible representations. One defines from φ a map $\hat{\varphi}$ on H with

$$\hat{\varphi}(h) : \hat{N} \longrightarrow \hat{N} , \quad \hat{\varphi}(h)[\hat{n}] \equiv [\hat{n}_h]$$

in the canonical way, i.e. by

$$\hat{n}_h(n) \stackrel{\text{def}}{=} \hat{n}(\varphi(h)^{-1}\cdot n) \tag{3.2}$$

The set of all classes $\{[\hat{n}_h]\}$ generated by $\hat{\varphi}(H)$ from a class $[\hat{n}]$ is called the orbit of $[\hat{n}]$ and will be denoted $O_{[\hat{n}]} \subseteq \hat{N}$. The action in (3.2) defines at the same time an <u>homogeneous little group</u> $H_{\hat{n}} \subseteq H$ by the invariance condition

$$h \in H_{\hat{n}} \quad \Longleftrightarrow \quad [\hat{n}_h] = [\hat{n}] \tag{3.3}$$

i.e. $h \in H_{\hat{n}}$ if and only if there exists a unitary operator $S(h)$ in $\mathcal{H}(\hat{n})$, the representation space of $\hat{n}$, such that

$$\hat{n}_h(n) = S(h)^{-1}\,\hat{n}(n)\,S(h) \quad , \quad \forall\, n \in N \quad . \tag{3.4}$$

The map $S : h \longmapsto S(h)$ is in general not a homomorphism, but can easily be shown from (3.4) (because of the irreducibility of $\hat{n}$ and Schur's Lemma) to be a projective map, satisfying thus

$$S(h_1)\,S(h_2) = \tau(h_1,h_2)\,S(h_1h_2) \tag{3.5}$$

for some factor set $\tau(h_1,h_2) \in U(1)$, the unit circle of the complex plane. The isotropy (or little) group of $[\hat{n}]$ $G_{\hat{n}} \subseteq G$ is defined as the subgroup of G that leaves $[\hat{n}]$ invariant under the action

$$g : \hat{n}(n) \longmapsto \hat{n}(g^{-1}ng)$$

N being identified with its image as a subgroup of G. This group $G_{\hat{n}}$ appears then as an extension (here trivial because so is G) of N by $H_{\hat{n}}$ as is shown in the following commutative diagram of exact sequences

$$\begin{tikzcd} 1 \arrow[r, dashed] & N \arrow[r] \arrow[d, equal] & G_{\hat{n}} \arrow[r, "\pi"] \arrow[d, "(\iota)"] & H_{\hat{n}} \arrow[r] \arrow[d, "(\iota)'"] & 1, & 0,\varphi \\ 1 \arrow[r] & N \arrow[r] & G \arrow[r, "\pi'"] & H \arrow[r] & 1, & 0,\varphi \end{tikzcd} \tag{3.6}$$

(ι) and $(\iota)'$ denoting the natural injection monomorphisms and π, π' the canonical epimorphisms. One can now construct, in a first step, a (projective) representation of $G_{\hat{n}}$ as follows: let L be a projective representation of $H_{\hat{n}}$ with factor set ω and carrier space $\mathcal{H}(L)$ and let $(\hat{n}_S L)$ be defined on $(n,h_{\hat{n}}) \in G_{\hat{n}}$ as follows

$$(\hat{n}_S.L)(n,h_{\hat{n}}) \overset{\text{def}}{=} \hat{n}(n)\ S(h_{\hat{n}}) \otimes L(h_{\hat{n}}) \qquad . \tag{3.7}$$

One obtains in this way a projective representation of $G_{\hat{n}}$ with factor set σ, where σ is obtained from (3.4), (3.5) and (3.7) to be equal to

$$\sigma(g_1,g_2) = \tau(\pi g_1,\pi g_2)\ \omega(\pi g_1,\pi g_2) \tag{3.8}$$

so that by choosing $\omega = \tau^{-1}$ (as we shall do in the sequel) one obtains an ordinary representation of $G_{\hat{n}}$ with carrier space $\mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$. The last step is now the following: one decomposes H in (right) cosets with respect to $H_{\hat{n}}$, with coset representatives $\{h_i | i \in I\}$, where I is some index set (Borel) isomorphic to $H/H_{\hat{n}}$ and similarly G with respect to $G_{\hat{n}}$, choosing now as coset representatives the images of $\{h_i\}$ under a fixed section $r : H \longrightarrow G$. This set of representatives is thus given by $\{(0,h_i) | i \in I\}$. Let now $\mu_{\hat{n}}$ be the (unique, as a class) invariant ergodic measure on $H/H_{\hat{n}}$, and let us also assume that this measure is right and left invariant (since $\bar{M}_o$ is unimodular, we assume thus G to be such). It follows then from the assumed regularity of (3.1) that this measure is also transitive (i.e. concentrated on the orbit). We may then identify the coset spaces $H/H_{\hat{n}}$ and $G/G_{\hat{n}}$ with $O_{[\hat{n}]}$

via the 1-to-1 Borel isomorphism $[\hat{n}_{h_i}] \longleftrightarrow h_i$, and $(0,h_i)$ respectively, so that we may use the same parametrization $\{h_i\}$ to describe all these spaces. Let us then consider a vector valued function $\varphi : \{h_i\} \longrightarrow \mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$ satisfying the 2 conditions

$$\begin{aligned} &(i) \quad (\varphi(h_i),\varphi') \text{ is } \mu_{\hat{n}}\text{- measurable}, \forall\ \varphi' \in \mathcal{H}(\hat{n}) \otimes \mathcal{H}(L) \\ &(ii) \quad \|\varphi\|^2 \equiv \int_I \|\varphi(h_i)\|^2 \, d\mu_{\hat{n}}(h_i) < \infty . \end{aligned} \tag{3.9}$$

These vector-valued functions span a separable Hilbert space [5] on which the induced representation is defined as follows, $\forall\ (n,h) \in G$

$$(\hat{n} \uparrow G)^L(n,h)\, \varphi(h_i) \stackrel{\text{def}}{=} (\hat{n}_S \cdot L)((0,h_i)(n,h)(0,h_j)^{-1})\varphi(h_j) \tag{3.10}$$

where h_j is the (unique) coset representative satisfying the condition $h_i\ h\ h_j^{-1} \in H_{\hat{n}}$. The following then holds, and is a consequence of [5] for this special case:

Theorem 3.1. (Mackey [5]). Consider an orbit $O_{[\hat{n}]}$ and a transitive ergodic measure $\mu_{\hat{n}}$ concentrated on it as described above. Then the representation (3.10) is unitary and irreducible if and only if L is. Two such representations $(\hat{n}_1 \uparrow G)^{L_1}$ and $(\hat{n}_2 \uparrow G)^{L_2}$ are equivalent if and only if $\hat{n}_1$ and $\hat{n}_2$ are in the same orbit and $L_1 \sim L_2$. Moreover all irreducible unitary representations of G are obtained in this way, up to equivalence, when one induces once per orbit and for each orbit one considers all inequivalent projective ω-representations L of the corresponding homogeneous little groups, with ω satisfying (3.8), for $\sigma = 1$, and τ determined by $\hat{n}$ through (3.4) and (3.5).

Let now N be nilpotent. It follows from theorem 2.1 and from (2.14) that in each class $[\hat{n}]$ we may choose as representative $V_{\nu,\underline{\ell}}$ defined as in (2.4) with $\underline{\ell}$ now ideal. Let $L = \exp \underline{\ell}$, then $L < N$ and we may construct the following exact sequence of groups:

$$1 \longrightarrow L \dashrightarrow N \xrightarrow{\pi} N/L \longrightarrow 1, \quad \rho,\psi \quad . \tag{3.11}$$

We have shown elsewhere [13] that in this case, the operators $S(h)$ and the factor set ω can be calculated explicitly and are respectively given by

$$\begin{aligned} S(h) &= V_{\nu,\underline{\ell}}(n(h)) \\ \omega(h_1,h_2) &= \rho(\pi n(h_1),\pi n(h_2)) \end{aligned} \tag{3.12}$$

where ρ is defined by (3.11) and $n(h)$ is the unique (up to L) element of N satisfying the condition

$$(\mathrm{coAd}(h)\cdot\nu)(\xi) = (\mathrm{coAd}(n(h))\cdot\nu)(\xi) \ , \quad \forall\ \xi \in \underline{\ell} \tag{3.13}$$

b) Induction to $\overline{M}_o$.

Let us now apply the general theory just mentioned to our group $\overline{M}_o$, which can, as seen in section 1, be written as a semi-direct product of K by $SL(2,\mathbb{C})$. The action (3.2) of $\overline{M}_o$, resp. $\overline{M}_o/K$ on K is obtained from (2.5) and (2.7) by the coadjoint action on the dual algebra $\underline{k}'$ of K, as shown in [13]. We obtain, for $\overline{m} = \langle B,a,\overline{\Lambda}\rangle \in \overline{M}_o$

$$\hat{\varphi}(\overline{m})(f,p) = (\Lambda^{-1}f,\ \Lambda^{-1}(p + a\cdot f)) \tag{3.14}$$

where the action of $\overline{\Lambda} \in SL(2,\mathbb{C})$ on p and f is given, via the covering map, by the action of the corresponding Lorentz transformation on covariant vectors and tensors respectively. For the homogenous part we obtain thus with $\overline{\Lambda} \in SL(2,\mathbb{C})$

$$\hat{\varphi}(\overline{\Lambda})(f,p) = (\Lambda^{-1}f,\ \Lambda^{-1}p) \tag{3.15}$$

Since a class of irreducible representations of K was characterized by an orbit $O_{(f,p)} = \{(f, p + a\cdot f)) | \forall a \in U\}$ the corresponding homogeneous little group is then given by the subgroup of $SL(2,\mathbb{C})$ which leaves the orbit invariant under (3.15), i.e. whose elements $\overline{\Lambda}$ fulfill the two conditions

$$\begin{aligned} &(i) \quad \Lambda^{-1} f = f \\ &(ii) \quad \Lambda^{-1} p = p + a' f , \quad \text{for some } a' \in U \end{aligned} \tag{3.16}$$

It is useful at this point, because of (3.16) (i) to parametrize f (as in [8]) as a formal constant uniform (c.u.) e.m. field with $f_{oi} = e_i$, $f_{ij} = \varepsilon_{ijk}\, b_k$, $i,j,k \in 1,2,3$ and ε_{ijk} the totally antisymmetric tensor of order 3, so that the first condition of (3.16) reduces to the well known problem of the symmetry group of such a field (see e.g. [12], [17] or [18]). These symmetry groups can be classified with the help of the following two Lorentz invariants of the tensor f:

$$\begin{aligned} i_1(f) &= (f\cdot f) \equiv f_{\mu\nu} f^{\mu\nu} = 2\,(\vec{b}^2 - \vec{e}^2) \\ i_2(f) &= (f\cdot f^*) \equiv f_{\mu\nu} (f^*)^{\mu\nu} = 4\,(\vec{b}\,.\,\vec{e}) \end{aligned} \tag{3.17}$$

where $(f^*)^{\mu\nu} = \frac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}$ is the dual of f ($\varepsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric tensor of order 4). It is easy to see that, because det f = $= (\vec{b}\,.\,\vec{e})^2$, the previous classification in cases (α) (β) and (γ) corresponds in this language to $f = 0$, $i_2(f) \neq 0$, and $i_2(f) = 0$, $f \neq 0$ respectively.

Before we determine the homogeneous little groups of the classes $[V_{(f,p)}]$ of irreducible representations of K, let us analyze more precisely what is the action of $SL(2,\mathbb{C})$ on them, for the three cases we had in turn. For this purpose, we can, as seen before, look equivalently at the action on the orbits $O_{(f,p)}$. This action follows from (3.15). Let us thus consider again the three cases separately

(α) The orbits O_p are mapped onto the collection $\{O_{\Lambda^{-1}p} | \forall\Lambda\in O_o(3,1)\}$,

(β) The orbits O_f are mapped onto the collection $\{O_{\Lambda^{-1}f} | \forall\Lambda\in O_o(3,1)\}$,

(γ) The orbits $O_{f,q}$ (with $q \in M^*(4)/\mathrm{Im}(f)$) are mapped onto the collection $\{O_{\Lambda^{-1}f,\Lambda^{-1}q}\}$. This follows from the fact that

$$\begin{aligned} \mathrm{Im}(\Lambda^{-1}f) &= \{a^\mu(\Lambda^{-1})^\rho_{\ \mu} (\Lambda^{-1})^\sigma_{\ \nu} f_{\rho\sigma} | \forall a \in U\} \\ &= \{(\Lambda^{-1})^\sigma_{\ \nu} (a')^\rho f_{\rho\sigma} | \forall a' = \Lambda a \in U\} = \Lambda^{-1}(\mathrm{Im}(f)) \end{aligned} \tag{3.18}$$

We shall denote in the sequel superorbits these sets (orbits of orbits) in order to avoid confusion. Let us remark here that all these superorbits are connected, since $O_o(3,1)$ is so [19]. Further, since the class of an induced representation of $\bar{M}_o$ does not depend on the element of the superorbit it is based on, we shall fix in each case the coordinates in such a way that the formal field f has a simple structure (note that this will imply that it will in general not be possible to put at the same time the other parameter $p \in M^*(4)$ in a form as simple as in the free field case). It is well known (see e.g. [12]) that for f as in case (β) it is always possible to choose a reference frame in such a way that $\vec{e} \parallel \vec{b} \parallel$ z-axis and for f as in case (γ) there are three possibilities depending on the first invariant of f in (3.17): if $i_1(f) > 0$, there exists a frame with $\vec{e} = 0$, and $\vec{b} \parallel$ z-axis; if $i_1(f) < 0$ there exists a frame with $\vec{b} = 0$, $\vec{e} \parallel$ z-axis, and if $i_1(f) = 0$ there exists a frame with $\vec{e} \parallel$ z-axis, $\vec{b} \parallel$ x-axis. We denote for obvious reasons (and as in [8]) these cases by (γ mag) (γ el) and (γ rad) respectively. With this choice of representatives we have made on the superorbits we have in fact also constructed, as is easily seen, a Borel cross-section, i.e. a Borel set in K^c intersecting per definition each superorbit once and only once. This ensures us that the semi-direct product of K by $SL(2,\mathbb{C})$ is regular in the sense of Mackey [5] and thus that all ergodic measures in the superorbits will be transitive so that any possible pathology is excluded (on the contrary of certain physically relevant subgroups of $\overline{M}_o$, see [20]). The theorem 2.1 may thus be applied.

With these particular choices on the superorbits, it is now an easy matter to determine the homogeneous little groups which are the subgroups of the symmetry groups of the various (formal) fields $f_{\mu\nu}$ fulfilling (3.16) (ii) too. This last condition may split of course in function of p each case in more subcases. In case (β) we know that p can always be chosen 0 so that (3.16) (ii) is always fulfilled. In case (γ) we have, with the above choices of coordinates, in $M^*(4)$

$$\begin{aligned}
(\mu \text{ mag}) &: Im(f) = \{(0,\lambda,\mu,0) \mid \forall\ \lambda,\mu \in \mathbb{R}\} \\
(\mu \text{ el }) &: Im(f) = \{(\lambda,0,0,\mu) \mid \forall\ \lambda,\mu \in \mathbb{R}\} \qquad (3.19)\\
(\mu \text{ rad}) &: Im(f) = \{(\lambda,0,\lambda,\mu) \mid \forall\ \lambda,\mu \in \mathbb{R}\}
\end{aligned}$$

so that the quotient spaces $M^*(4)/Im(f)$ can be parametrized by (p_o,p_3), (p_1,p_2), and $(p_1,\sqrt{\tfrac{1}{2}}(p_o-p_2) \overset{\text{def}}{=} p_-)$ respectively. We do not go into details of all these calculations, since they are tedious but straightforward: we just give the complete results in the form of Tables. The homogeneous little groups are listed in Table I where the elements of $SL(2,\mathbb{C})$ are characterized by their generators in $s\ell(2,\mathbb{C})$. We also already include in this Table the elements of the Poincaré group which are not in the connected component and whose action is defined in the next section on p and f. We denote these last elements m_i for a mirror perpendicular to the i-axis; an accent means time-inversion and a bar space-inversion. For the case (α) we just include the well known results of Wigner for completeness (see e.g. [19], [21]). In Table II we list other useful results for each case, in particular the isomorphism classes of the homogeneous little groups (including again discrete symmetries and denoting C_2' a discrete generator conof gererators of the corresponding coset decompositions. Finally ⊗ denotes a direct product and Ⓢ a semidirect product.

TABLE I

Homogeneous little groups of the representants $0_{(f,p)}$ of the superorbits

Case	Subcases	Infinitesimal generators	Discrete transformations unitary	Discrete transformations antiunitary
(α)	(i) $p=p_o(1,0,0,0), p_o\neq 0$	M_{12}, M_{13}, M_{23}	$\overline{1}$	$1'$
	(ii) $p=p_o(1,0,1,0), p_o\neq 0$	$M_{13}, M_{03}+M_{23}, M_{12}-M_{01}$	m_1	m_2'
	(iii) $p=p\ (0,1,0,0)$	M_{23}, M_{02}, M_{03}	m_2	$\overline{1}'$
	(iv) $p=0$	$M_{\mu\nu}$, $\forall \mu,\nu \in 0,..,3$	$\overline{1}$	$1'$
(β)	$p = 0$	M_{12}, M_{03}		m_1'
(γm)	(i) $p_3 \neq 0$	M_{12}		
	(ii) $p_3 = 0 \neq p_o$	M_{12}	$\overline{1}$	m_1'
	(iii) $p_3 = p_o = 0$	M_{12}, M_{03}	$\overline{1}$	m_1'
(γe)	(i) $p_1 \neq 0$ or $p_2 \neq 0$	M_{03}	$m_{(-2,1)}$	
	(ii) $p_1 = p_2 = 0$	M_{03}, M_{12}	m_1	$\overline{1}'$
(γr)	(i) $p_1 \neq 0 \neq p_-$	$M_{03} + M_{23}$		
	(ii) $p_1 = 0 \neq p_-$	$M_{03} + M_{23}$	m_1	m_2'
	(iii) $p_1 \neq 0 = p_-$	$M_{03}+ M_{23}$, $M_{12}- M_{01}$		
	(iv) $p_1 = 0 = p_-$	$M_{03}+ M_{23}$, $M_{12}- M_{01}$	m_1	m_2'

TABLE II

Isomorphy classes of homogeneous little groups and coset spaces

Case	Isomorphy classes	Coset spaces: Dimension	Coset spaces: parameter spaces	Coset spaces: coset representants
(α)	(i) $SU(2) \otimes (C_2 \otimes C'_2)$	3	$\mathcal{R}^3$	M_{01}, M_{02}, M_{03}
	(ii) $\Delta(2) \otimes (C_2 \otimes C'_2)$ 1)	3	$\mathcal{R}^3$	M_{12}, M_{23}, M_{02}
	(iii) $SL(2,\mathcal{R}) \circledS (C_2 \otimes C'_2)$	3	$\mathcal{R}^3$	M_{01}, M_{12}, M_{13}
	(iv) $SL(2,\mathcal{C}) \circledS (C_2 \otimes C'_2)$	0	-	-
(β)	$(\mathcal{R}/4\pi\mathbb{Z} \otimes \mathcal{R}) \circledS C'_2$	4	$\mathcal{R}^4 \times 2$	$M_{01}, M_{02}, M_{13}, M_{23}, \bar{1}$
(γm)	(i) $\mathcal{R}/4\pi\mathbb{Z}$	5	$\mathcal{R}^5 \times 4$	$M_{01}, M_{02}, M_{03}, M_{12}, M_{23}, \bar{1}, 1'$
	(ii) $\mathcal{R}/4\pi\mathbb{Z} \circledS (C_2 \otimes C'_2)$	5	$\mathcal{R}^5$	$M_{01}, M_{02}, M_{03}, M_{12}, M_{23}$
	(iii) $(\mathcal{R}/4\pi\mathbb{Z} \otimes \mathcal{R}) \circledS (C_2 \otimes C'_2)$	4	$\mathcal{R}^4$	$M_{01}, M_{02}, M_{13}, M_{23}$
(γe)	(i) $\mathcal{R} \otimes C_2$	5	$\mathcal{R}^5 \times 2$	$M_{01}, M_{02}, M_{12}, M_{13}, M_{23}, \bar{1}'$
	(ii) $(\mathcal{R}/4\pi\mathbb{Z} \otimes \mathcal{R}) \circledS (C_2 \otimes C'_2)$	4	$\mathcal{R}^4$	$M_{01}, M_{02}, M_{13}, M_{23}$
(γr)	(i) $\mathcal{R}$	5	$\mathcal{R}^5 \times 4$	$M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, \bar{1}, 1'$
	(ii) $\mathcal{R} \circledS (C_2 \otimes C'_2)$	5	$\mathcal{R}^5$	$M_{01}, M_{02}, M_{03}, M_{12}, M_{13}$
	(iii) $\mathcal{R}^2$	4	$\mathcal{R}^4 \times 4$	$M_{02}, M_{12}, M_{13}, M_{23}, \bar{1}, 1'$
	(iv) $\mathcal{R}^2 \circledS (C_2 \otimes C'_2)$	4	$\mathcal{R}^4$	$M_{02}, M_{12}, M_{13}, M_{23}$

1) For this notation, see [19].

The last step of the induction procedure to $\overline{M}_o$, as described in the first part of this section, is now the following: from each of the superorbits one takes the irreducible representation $V_{(f,p)}$ of K as given by (2.17), (2.18) or (2.19), with (f,p) as just chosen and one constructs all unitary irreducible representations of $\overline{M}_o$ as in (3.10). The result is then as follows

$$(V_{(f,p)\cdot S} \uparrow \overline{M}_o)^L \langle B,a,\overline{\Lambda}\rangle\, \varphi(\overline{\Lambda}_i) =$$
$$= (V_{(f,p)\cdot S} \cdot L)(\langle 0,0,\overline{\Lambda}_i\rangle\langle B,a,\overline{\Lambda}\rangle\langle 0,0,\overline{\Lambda}_j^{-1}\rangle)\, \varphi(\overline{\Lambda}_j) \qquad (3.20)$$

with again the usual identifications, with $\overline{\Lambda}_j$ determined by the condition that $\overline{\Lambda}_i\ \overline{\Lambda}\ \overline{\Lambda}_j^{-1}$ is in the homogeneous little group of $V_{(f,p)}$ and with $\varphi(\overline{\Lambda}_i)$ defined as in (3.9). In (3.20), the representation $(V_{(f,p)\cdot S}\cdot L)$ of the little group (here the semidirect product of K by the corresponding homogeneous little group) is given, as in (3.7), by

$$(V_{(f,p)\cdot S}\cdot L)\,\langle B,a,\overline{\Lambda}\rangle = V_{(f,p)}(\langle B,a,1\rangle)\, S(\overline{\Lambda}) \otimes L(\overline{\Lambda}) \qquad (3.21)$$

for some ω-representation L of the homogeneous little group, with ω as in (3.12), and which has now to be calculated explicitly. We first observe therefore in Table I that, except in case (α) of course, all homogeneous little groups are abelian (we consider now again the connected component of $\overline{M}$ to unity only), and their Lie-algebra is isomorphic either to $\mathbb{R}$ or to $\mathbb{R}^2$, so that, using the well known result of Bargmann [7] on the relevant cohomology groups

$$H^2(\mathbb{R}^n, U(1)) \cong \mathbb{R}^{n(n-1)/2} \qquad (3.22)$$

we know already that only in the two dimensional cases the factor set ω could be non-trivial. Let us now use the more precise result (3.12) for the induction from a normal nilpotent subgroup: it shows that ω, as a class, is also an element of $H^2(N/L, U(1))$, since ρ is a factor set on N/L and $T_{(f,p)}$ is unitary and one-dimensional (in fact $[\omega] \in H^2(N/L(H_{\hat{n}}), U(1))$ where $N/L(H_{\hat{n}})$ is the subgroup of N/L generated by the classes of all $n(h)$, $h \in H_{\hat{n}}$). Let us now then consider again the various cases in turn, with now $N = K$:

Case (α) $K/L \cong 1$ (from (2.14)) so that $\omega = 1$. This is of course trivial and well know, but it shows how our result works.

Case (β) $K/L \cong \mathbb{R}^2$ (from (2.14)) thus non-trivial multipliers could occur. The elements $n(h)$ of (3.12) are given by the translations a' in (3.16) (ii) and thus the multiplier ω by $T_{(f,p)}(A(a_1', a_2'))$ where A is the factor system (1.3) as restricted to $U \times U$. We had however seen that, choosing $p = 0$ (as it is always possible in this case) $a' = 0$ satisfies (3.16) (ii) $\forall\, \bar{\Lambda} \in SL(2,\mathbb{C})$, so that ω is necessarily trivial, A being then equal to zero.

Case (γ) $K/L \cong \mathbb{R}$ (from (2.15)) thus by (3.22) ω is, also in this case, trivial.

These results make the situation quite easier, because we then only have to consider ordinary representations of the homogeneous little groups and these are of so simple structure (except in case (α) they are all abelian) that this problem is easily solved: the irreducible unitary representations of $\mathbb{R}/4\pi\mathbb{Z}$ are given by $\{e^{ijr}, r \in \mathbb{R}/4\pi\mathbb{Z}\ , 2j \in \mathbb{Z}\}$ and the ones of $\mathbb{R}$ by $\{e^{i\lambda r}, r \in \mathbb{R}, \lambda \in \mathbb{R}\}$. We shall call spins, similarly as in the free case, the labels j resp. λ of these representations and spinors the square μ-integrable functions on the corresponding coset spaces, with μ the ergodic (here transitive) measures on this spaces under the action of $SL(2,\mathbb{C})$. A complete list of these spins and of the dimension (i.e. number of independent components) of the corresponding spinors will be given at the end of the next section and we shall discuss in section 5 the physical meaning of these spins.

We first want to reintroduce the discrete transformations related to space and time reversals, too.

4. Inclusion of the discrete transformations.

So far we have obtained all projective unitary representations of the connected component M_o of M (by means of the ordinary unitary irreducibles of $\overline{M}_o$). If we want to reintroduce the discrete transformations and calculate all PUAIR of M we first have to calculate all multipliers (1.7) on M, with respect to the UA decomposition $M^{\uparrow} \cup M^{\downarrow}$, ($M^{\uparrow}$ being, as seen in section 1, the subgroup of M corresponding to orthochronous Poincaré transformations only). Let us therefore consider the following exact sequence of groups

$$1 \longrightarrow \overline{M}_o \longrightarrow \overline{M} \longrightarrow V_4 \longrightarrow 1, \quad \varphi \qquad (4.1)$$

where V_4 is the Klein Vierergruppe. Since M is a split extension of K by O(3,1) (see section 1), V_4 can be identified with the subgroup of M (and of $\overline{M}$) generated by $<0,0,1>$, $<0,0,1'>$, $<0,0,\overline{1}>$ and $<0,0,\overline{1}'>$ thus (4.1) is also split. The action φ of V_4 on $\overline{M}_o$ follows then from (1.2) and from the corresponding action of the Poincaré group. Using (4.1) we may write each element of $\overline{M}$ as a pair $(\overline{m},h)$ with $\overline{m} \in \overline{M}_o$ and $h \in V_4$. The product reads then

$$(\overline{m},h) . (\overline{m}',h') = (\overline{m} . \varphi(h)\overline{m}',hh')$$

The problem of the multipliers on M can now be solved, using the following:

<u>Proposition 4.1.</u> Let σ be a multiplier on $\overline{M} \times \overline{M}$ (w.r. to the above UA decomposition), then there exists a multiplier $\sigma_1 \sim \sigma$ with

$$\sigma_1((\overline{m},h)\,,\,(\overline{m}',h')) \;=\; \delta(h,h') \qquad (4.2)$$

where δ is a multiplier on $V_4 \times V_4$ (w.r. to the UA decomposition $\{1,\overline{1}\} \cup \{1',\overline{1}'\}$).

<u>Proof</u>: Since $\overline{M}$ is a semidirect product in (4.1) the same argumentation as in theorem 9.4. of [5b] applies (see also [6] for this extension of Mackey's proof) so that $\sigma \sim \sigma_1$ with

$$\sigma_1((\overline{m},h)\,,\,(\overline{m}',h')) \;=\; \tau(\overline{m},\varphi(h)\overline{m}')\;\delta(h,h')\;\psi(\overline{m}',h) \qquad (4.3)$$

where ψ is a Borel function from $\overline{M}_o \times V_4$ on $U(1)$, τ and δ are multipliers on $\overline{M}_o \times \overline{M}_o$ and $V_4 \times V_4$ (w.r. to the given UA decomposition) respectively, and τ and ψ satisfy, $\forall\ \overline{m},\overline{m}' \in \overline{M}_o$, $h,h' \in V_4$

$$\begin{aligned} &\text{(i)}\ \ \tau(\varphi(h)\overline{m},\varphi(h)\overline{m}') \;=\; \tau(\overline{m},\overline{m}')\psi(\overline{m}\,\overline{m}',h)\psi(\overline{m},h)^{-1}\psi(\overline{m}',h)^{-1} \\ &\text{(ii)}\ \ \psi(\overline{m},hh') \;=\; \psi(\varphi(h')\overline{m},h)\;\psi(m,h') \end{aligned} \qquad (4.4)$$

Since any multiplier on $\overline{M}_o \times \overline{M}_o$ is trivial, τ is necessarily so. Using this fact and the equations (4.4) it is straightforward to verify that τ may be then chosen equal to one, as giving rise to an equivalent factor system σ_1 in (4.3). It follows then from (4.4) (i) that for any h, ψ is a 1-dimensional unitary representation of $\overline{M}_o$ and because the only 1-dimensional unitary representation of $\overline{M}_o$ is the identity (from our previous results), $\psi \equiv 1$. The proof follows then from (4.3).

The multipliers on $V_4 \times V_4$ (w.r. to the given UA decomposition) are well known (see e.g. [7]). There are 4 inequivalent classes with representants $\delta^{\alpha\beta}$, where $\alpha,\beta = \pm 1$ and

$\delta^{\alpha\beta}$	1	$\overline{1}$	$\overline{1}'$	$1'$
1	1	1	1	1
$\overline{1}$	1	1	1	1
$\overline{1}'$	1	$\sigma\beta$	α	β
$1'$	1	$\alpha\beta$	α	β

(4.5)

The last step for obtaining all PUAIP of $\mathbf{M}$ would thus now be to induce the unitary representations of $\overline{M}_o$ to $\overline{M}$. The theory of Mackey has for this purpose to be slightly generalized in order to take the antiunitary character into account. This generalization has recently been achieved by Shaw and Lever [22] and we refer to their paper for a detailed description of the theory, whose application is quite straightforward in our problem and will therefore just briefly be sketched here.

Let us first determine the action of the discrete transformations on the dual $\hat{\overline{M}}_o$ of $\overline{M}_o$. This action is obviously given by the corresponding action on an element (f,p) of the dual algebra $\underline{k}'$ of $\underline{k}$, and similarly as in the free particle case, we assume that elements of $\mathbf{M}$ containing time-reversal are represented by antiunitary operators. We obtain in this way

$$\hat{\varphi}(h)\cdot p = \varepsilon(h)\ \Lambda(h)^{-1}p$$
$$\hat{\varphi}(h)\cdot f = \varepsilon(h)\ \Lambda(h)^{-1}f \qquad (4.6)$$

with $h \in V_4$, $\Lambda(h)$ the corresponding Lorentz transformation and

$$\varepsilon(h) = \text{sign}\ (\Lambda^o_o(h)) \quad .$$

This action determines, for any irreducible representation of $\overline{M}_o$, and similarly as before, little groups as the groups that leave this representation (as a class) invariant. Because of the unitary/antiunitary character of the representations, these groups are termed _generalized little groups_ (respectively _generalized homogeneous little groups_). Using the action in (4.6), and the explicit form of the representations we have found previously, these generalized little groups can be straightforwardly computed. The results are listed in Table I.

Once these groups are known one may further induce the corresponding representations to the whole of **M**. Since this procedure of "generalized inducing" has already been applied [23] for the PUAIR of the Poincaré group (our case (α)), we indicate briefly for the case (β), and for illustration, the essential steps of the procedure:

In case (β), the generalized homogeneous little group G is given (see Tables I and II) by $G = (\mathcal{R}/4\pi\mathbb{Z} \otimes \mathcal{R}) \circledS C_2'$ with generators M_{12}, M_{03} and $h = \pi_1'$ respectively. The UA decomposition of G reads then

$$G \;=\; G^+ \cup G^- \;=\; \{M_{12}, M_{03}\} \cup \pi_1' \{M_{12}, M_{03}\} \qquad . \qquad (4.7)$$

The unitary irreducible representations $D^{j,\lambda}$ of G^+ are 1-dimensional and labelled by (j,λ) with $2j \in \mathbb{Z}$ and $\lambda \in \mathcal{R}$. Since h is in the same connected component as $\overline{T}'$ we have by (4.2) and (4.5)

$$\sigma^{\alpha\beta}((0,h),\,(0,h'\,)) \;=\; \delta^{\alpha\beta}(\overline{T}',\overline{T}') \;=\; \alpha \qquad .$$

We now, using theorem B of [22], define the following $\overline{\sigma}$-representation of G^+, with $\overline{\sigma}$ the factor system obtained from σ by complex conjugation:

$$E^{j,\lambda}(g) \;=\; \overline{\sigma(g,h)}/\overline{\sigma(h,\varphi(h)^{-1}g)}\; D^{j,\lambda}(\varphi(h)^{-1}g) \qquad (4.8)$$

with $g \in G^+$ and $\sigma(g,h)$ a shorthand notation for $\sigma((\overline{m},1)\,,(1,h))$. The following possibilities can now occur:

if E and D are antiunitarily equivalent (in the sense of ordinary (and not projective) equivalence) by means of an antiunitary operator **K**, then the induced PUAIR will be of Wigner-Type I respectively of Wigner-Type II if $K^2 = \sigma(h,h)\ D(h^2)$, and respectively $K^2 = -\sigma(h,h)\ D(h^2)$. If no such **K** exists then it will be of Wigner-Type III [1].

Using now the commutation relations

$$\begin{aligned} \Lambda(m_1')\ \Lambda(M_{03}) &= \Lambda(-M_{03})\ \Lambda(m_1') \\ \Lambda(m_1')\ \Lambda(M_{12}) &= \Lambda(-M_{12})\ \Lambda(m_1') \end{aligned} \tag{4.9}$$

we get for (4.8)

$$E^{j,\lambda}(g) = D^{j,\lambda}(g^{-1}) = D^{-j,-\lambda}(g) \quad . \tag{4.10}$$

Hence the representations E and D are antiunitarity equivalent by means of the complex conjugation operator **K**. As $K^2 = 1$ the induced PUAIR will be, from the criterion just given, of (Wigner-) Type I if $\alpha = (-)^{2j}$ ($K^2 = \delta^{\sigma f}(h,h)\ D^{j,\lambda}(h^2)$ with $D^{j,\lambda}(h^2) = D^{j,\lambda}((m_1')^2) = (-)^{2j}$) and of (Wigner-) Type II if $\alpha = (-)^{2j+1}$. In the first case there is no doubling of states and the restriction of the inducing representation on G^+ is irreducible. In the second case the carrier space is doubled and the representation is given by

$$U^{j,\lambda}(g) = \begin{pmatrix} D^{j,\lambda}(g) & 0 \\ 0 & D^{j,\lambda}(g) \end{pmatrix} \quad U^{j,\lambda}(m_1') = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \tag{4.11}$$

the restriction to G^+ being thus then reducible in two equivalent

---

1) The (Wigner) Type we refer to have of course nothing to do with the (Murray-von Neumann) Types we referred to previously.

irreducible subrepresentations. Taking into account the doubling of states occuring from the fact that the coset-space of the complete superorbits has two connected components we get finally, inserting also the dimension of the corresponding spinors, the following possibilities

	α	β	Wigner Type	dimension inducing representation	dimension spinors
Case (β)	$(-)^{2j}$	$\pm$	I	1	2
	$(-)^{2j+1}$	$\pm$	II	2	4

(4.12)

Denoting then $\Delta^{j,\lambda}_{(\beta)}$ the representation of $\bar{M}_c$ based on an orbit $O_{(f,p)}$ with det $f \neq 0$ and with spin-values j and λ, the whole PUAIR of $\bar{M}$ (and hence, via the covering map of SL(2,$\mathbf{C}$) onto $O_o(3,1)$), the whole PUAIR of M is finally given by

$$(\Delta^{j,\lambda}_{(\beta)} \uparrow M)\ (\bar{m},n)\ \varphi_r(\bar{\Lambda}_i) = \tag{4.13}$$
$$= \frac{\delta^{\alpha\beta}(h,r)}{\delta^{\alpha\beta}(s,rhs^{-1})}\ (V_{(f,p)}\cdot S\cdot U^{j,\lambda})(\bar{\Lambda}_{i,r}(\bar{m},h)\bar{\Lambda}^{-1}_{j,s})\ \varphi_s(\bar{\Lambda}_j)$$

where r,s label the different connected components of the complete superorbits (and the corresponding (discrete) elements of V_4), $\bar{\Lambda}_i$, $\bar{\Lambda}_j$ are as before, with the usual identifications, $\bar{\Lambda}_{i,r}(\bar{m},h)\bar{\Lambda}^{-1}_{j,s} \in G$, the generalized homogeneous little group (as e.g. in (4.7)) and $(V_{(f,p)}\cdot S\cdot U^{j,\lambda})$ is given as in (3.21) with I replaced by U (this last representation being then as constructed explicitely in (1.11) for the illustrative example we have discussed).

More important than this formula (which essentially illustrates how the factor set and the antiunitary operators are introduced in the generalized induction procedure), are of course the characterizations of the various PUAIR of M by means of the spins and of the dimensions of the corresponding spinors. The latter are defined, in analogy with the free particle case, by the number of independent states with definite $f_{\mu\nu}$ and p_{μ} (modulo Im(f)) in an irreducible representation. In order words, calling states the carrier space functions (i.e. the μ-measurable square integrable functions on the superorbits as in (3.9) with values in the carrier space of the representations U of the generalized homogeneous little groups G), the dimension of the spinors is then given by the dimension of this representation of G times the number of connected components in the superorbit.

The case (γ) splits in more subcases (see Tables I and II) but does not present very different situations than the one we have just sketched. Since the calculations are quite straightforward applications of the general theory, we again drop the details and list only the results (Table III). We include for completeness in this Table the free particle case, too (as obtained in [23]),but for some "physical" representations only.

As it can be seen from this Table, as soon as an external e.m. field (with non-zero c.u. part, see [1] and section 5) is present, the dimension of the spinors, as well as the spin-labels, may change discontinuously, together in fact with the discontinuous change which has occured in the representation class of the covariance group. Another fact, at first sight perhaps surprising, is that one or more continuous spins occur for some PUAIR of the new covariance group M. We remark however that these continuous spins are of quite a different type than the well known continuous spins of $m^2 \leqslant 0$ free particles, since the corresponding spinors have here only finitely many components. We also remark that in certain cases, there are no longer halfinteger spins to characterize the representations, this corresponding of course to the fact that, in these cases and because of the presence of the external field, there is no longer any symmetry transformation which is (conjugated to) a rotation in space.

TABLE III

Characterization of the PUAIR of M

Case	Spins of the PUAIR	Subcases	Factor sets α	Factor sets β	Wigner Type	Dimension repr. G	Dimension spinors
(α) (i)	j		$(-)^{2j}$	$(-)^{2j}$	I	$2j+1$	$2j+1$
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	$2(2j+1)$	$2(2j+1)$
			$\pm$	$\mp$	III	$2(2j+1)$	$2(2j+1)$
(ii)	$\lambda_1, \lambda_2, \pm j$	$\lambda_1=\lambda_2=0$, $j \neq 0$	$\pm(-)^{2j}$	$\pm(-)^{2j}$	I	2	2
			$\pm$	$\mp$	II	4	4
		$\lambda_1=\lambda_2=0$, $j = 0$	$(-)^{2j}$	$(-)^{2j}$	I	1	1
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	2	2
			$\pm$	$\mp$	III	2	2
		else	$\pm$	$\pm$, $\mp$		∞	∞
(β)	$\lambda, \pm j$		$(-)^{2j}$	$\pm$	I	1	2
			$(-)^{2j+1}$	$\pm$	II	2	4
(γm) (i)	$\pm j$		$\pm$	$\pm$, $\mp$	I	1	4
(ii)	$\pm j$		$(-)^{2j}$	$(-)^{2j}$	I	1	1
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	2	2
			$\pm$	$\mp$	III	2	2

(see continuation next page)

TABLE III (continued)

Case	Spins of the PUAIR	Subcases	Factor sets α	Factor sets β	Wigner Type	Dimension repr. G	Dimension spinors
(γm)(iii)	$\lambda, \pm j$	$\lambda = 0$	$(-)^{2j}$	$(-)^{2j}$	I	1	1
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	2	2
			$\pm$	$\mp$	III	2	2
		$\lambda \neq 0$	$(-)^{2j}$	$(-)^{2j}$	I	2	2
			$(-)^{2j+1}$	$(-)^{2j+1}$	II	4	4
			$\pm$	$\mp$	III	4	4
(γe)(i)	λ		$\pm$	$\pm, \mp$	I	1	2
(ii)	$\lambda, \pm j$	$j \neq 0$	$\pm$	$\pm, \mp$	III	4	4
		$j = 0$	$\pm$	$\pm, \mp$	III	2	2
(γr)(i)	λ		$\pm$	$\pm, \mp$	I	1	4
(ii)	λ		$+$	$+$	I	1	1
			$-$	$-$	II	2	2
			$\pm$	$\mp$	III	2	2
(iii)	λ_1, λ_2		$\pm$	$\pm, \mp$	I	1	4
(iv)	λ_1, λ_2		$+$	$+$	I	2	2
			$-$	$-$	II	4	4
			$\pm$	$\mp$	III	4	4

We are thus naturally lead to a first conclusion: except in case (α), the (usual) concept of spin has, as a matter of fact,lost the group theoretical meaning it had in the free case and, as a consequence, it is no longer necessarily related to the number of independent states or to a characterization of these states. However, new spin label(s) appear in a natural way as a consequence of a covariance principle in the actual situation, in presence thus of an external e.m. field, and as a characterization of representations of the new covariance group. Consider for illustration the case of a c.u. magnetic field: the corresponding representations are still characterized by a discrete half-integer spin label that has however a different and more profound significance than simply a kind of "remembering" of the (free) spin as transposed in the new situation. This spin has an intrinsic new meaning, and is then of "helicity" type, i.e. corresponds to only two polarization states (see Table III), not along an arbitrary axis as in the free case, but along the magnetic field axis (this corresponding in fact to what is physically observed).

In this respect, the external approximation is conceptually more far reaching than perhaps expected and it can be useful to extend the (group-theoretical) definition of an elementary particle, as adapted to this new situation and as in a new "world" where the external field is present. This is what we shall do in the next section, together with a discussion of our results and of some, we think important, consequences of them.

5. Discussion.

a) elementary particles in external e.m. fields.

A real (physical) particle can be identified, as a set of states, with the solutions of some covariant equation of motion, i.e. with a representation space of the general covariance group. Since the external field is an approximation, it is however not necessary that the corresponding representation is irreducible. Nevertheless it is useful to introduce the following group theoretical

Definition "An elementary (relativistic) particle in an external e.m. field is a quantum mechanical system which, as a set of states, spans the carrier space of an irreducible PUAIR of the general covariance group M".

It is clear from (1.5) and [1] that when the c.u. part of the external field is zero this definition reduces to the well known group theoretical definition of relativistic free particles 1) of Wigner. Note however that even then the covariance operator group may be only isomorphic to the ordinary Poincaré action (see e.g. below the formulas (5.5) or (5.6) with compensating gauges χ_g possibly non zero). This shows incidentally that for a given representation the concrete, physical, representation space is far from being unique in the sense that physically inequivalent representations do occur which are mathematically equivalent: free particles and, say, particles in an external crystal or radiation field, may in fact correspond to mathematically equivalent representations of M.

---

1) By this definition we clearly leave open the problems of the physical interpretation and existence of the discrete symmetry operators [24]. We assume here that they represent exact and rigourously valid symmetries of space-time.

As a consequence of its definition, an elementary particle in an external e.m. field may be characterized by the labels of the corresponding representation, i.e. by its spins (see Table III), and, in addition, by the values of the invariants of the Lie-algebra of $\mathbb{M}$ in this representation. These invariants can be obtained from the Lie-algebra (1.9) (see [12]) and are given explicitly by

$$
\begin{aligned}
Q_1 &= \mathbb{F}_{\mu\nu} \cdot \mathbb{F}^{\mu\nu} \\
Q_2 &= \mathbb{F}_{\mu\nu}(\mathbb{F}^*)^{\mu\nu} \\
Q_3 &= \Pi_\mu \Pi^\mu - M_{\mu\nu}\mathbb{F}^{\mu\nu} \qquad\qquad (5.1) \\
Q_4 &= 2\,\Pi_\mu(\mathbb{F}^*)^{\mu\nu}\,\Pi^\rho(\mathbb{F}^*)_{\rho\nu} - (\mathbb{F}^*)^{\mu\nu}M_{\mu\nu} \cdot Q_2
\end{aligned}
$$

where $(\mathbb{F}^*)^{\mu\nu}$ is the "dual" of $\mathbb{F}_{\mu\nu}$, defined in a similar way as in (3.17) for a field, i.e.

$$
(\mathbb{F}^*)^{\mu\nu} \equiv \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathbb{F}_{\rho\sigma} \qquad .
$$

A basis element of the carrier space of a PUAIR of $\mathbb{M}$ being as we saw characterized by an element $(f,p) \in \underline{k}'$ on some superorbit and by a set $\{s_i\}$ of spins, we denote it $|f,p,s_i\rangle$. It follows from (4.13), with (2.15), that in all representations we have

$$
\mathbb{F}_{\mu\nu}|f,p,s_i\rangle = f_{\mu\nu}|f,p,s_i\rangle \qquad\qquad (5.2)
$$

so that, comparing with (1.5), we see that the eigenvalues of these operators are equal (up to a factor $-\frac{e}{c\hbar}$) to the c.u. part of the field, as expected. The interpretation of the first two invariants in (5.1) is then quite obvious: they specify so to say the "world" in which the corresponding particle is "elementary", by means of the invariants of the corresponding (c.u. part of the) field. The two last invariants Q_3 and Q_4 correspond then to the particle itself and will thus be related to the equation of motion. Let us note here that we can now explain why we do not share the point of view in [12] by which the Lie-algebra (1.9) is rejected as "non-informative": these authors observe that this Lie-algebra gives rise to too many invariants in (5.1) in order to characterize the degrees of freedom of an elementary particle, because they want to characterize such a particle in terms of its free quantum numbers, whereas these are in our opinion no longer a good characterization of the states of that particle, as a consequence of the presence of the external field and thus as a consequence of the change occured in the covariance.

b) covariant equations of motion. Irreducible PUA representations of symmetry groups describe the properties of solutions, i.e. give informations on the classification of the possible states of a quantum-mechanical system and on their properties (invariants of these states, matrix elements, selection rules and so on). Irreducible PUA representations of covariance groups describe properties of the equations of motion. Let us therefore consider first the Klein-Gordon and the Dirac operators, with minimal coupling:

$$O_{KG}(x,\pi F) = (\hat{p}_\mu - \frac{e}{c}(\pi F)_\mu(x))^2 - m^2c^2 \qquad (5.3)$$

$$O_D(x,\pi F) = -i\gamma^\mu(\hat{p}_\mu - \frac{e}{c}(\pi F)_\mu(x)) - mc \qquad (5.4)$$

where $\hat{p}_\mu = -i\hbar\partial_\mu$ and πF is some uniquely chosen potential (as fixed by a map π). Remember that, as we had shown in [1], the choice of π is not essential. Let us then consider the following operators, $\forall\, m = \langle B,g\rangle \in \mathbf{M},\ B \in \mathbf{B},\ g \in IO(3,1)$

$$V_{KG}(\langle B,g\rangle) \overset{\text{def}}{=} \exp\{-i\frac{e}{c\hbar}((B\cdot gF^{(o)}) + \chi_g(\pi(gF),x))\}\ C_g\cdot P_g \quad (5.5)$$

and

$$V_D\ (\langle B,g\rangle) \overset{\text{def}}{=} \exp\{-i\frac{e}{c\hbar}((B\cdot gF^{(o)}) + \chi_g(\pi(gF),x))\}\ S_g\cdot P_g \quad (5.6)$$

where $\chi_g(\pi(gF),x)$ are the compensating gauge functions as for example obtained in (3.32) of [1], $F^{(o)}$ is the c.u. part of the field (see (3.24)-(3.28) of [1]), P_g is the substitution operator in the x-coordinate. Furthermore S_g is given by $S_g^o \cdot C_g$ where

$$C_g = \begin{cases} 1 & \text{for } g \text{ orthochronous} \\ C & \text{for } g \text{ antichronous} \end{cases} \quad (5.7)$$

with C the charge conjugation operator and S_g^o is determined by the condition that for Λ the homogeneous part of g we have

$$(S_g^o)^{-1}\ \gamma^\mu\ S_g^o = \Lambda^\mu_\nu\ \gamma^\nu \quad . \quad (5.8)$$

A straightforward calculation, similar as in [25], shows now that for the Klein-Gordon operator we have

$$\begin{aligned} &V_{KG}(\langle B,g\rangle)\ O_{KG}(x,\pi F)(V_{KG}(\langle B,g\rangle))^{-1} \\ &= O_{KG}(x,g\pi F - \partial\chi_g(\pi(gF),x)) \quad (5.9) \\ &= O_{KG}(x,\pi(gF)) \end{aligned}$$

by definition of the gauge transformation $\chi_g(\pi(gF),x)$. Similarly, for the Dirac operator we obtain with (5.6)

$$V_D(\langle B,g\rangle)\ O_D(x,\pi F)(V_D(\langle B,g\rangle))^{-1} = O_D(x,\pi(gF)) \quad . \tag{5.10}$$

This means that, by definition of the covariance in the presence of an external e.m. field (see section 4 of [1]), both the Klein-Gordon and the Dirac equations are covariant under M, as expected, and for the whole class of (almost) arbitrary external fields we have considered. On the other side, it may be explicitly verified that the operators defined in (5.5) and in (5.6) are homomorphic on $\bar{M}$, i.e. generate (projective) representations of M. With the usual definitions of the scalar products of the free equations of motion in these both cases, these representations are also obviously unitary (on $M^{\dagger}$).

As follows then also from the above equations (5.9) and (5.10) if $\psi(x,F)$ is a solution of the Klein-Gordon or of the Dirac equation in presence of the field F, i.e. satisfies for the respective operator (5.3) or (5.4) the equation

$$O(x,\pi F)\ \psi(x,F) = 0 \tag{5.11}$$

we have, with $\Phi(x,gF) \equiv (V\langle B,g\rangle\psi)(x,F)$ as given by (5.5), respectively (5.6):

$$O(x,\pi(gF))\ \Phi(x,gF) = 0 \tag{5.12}$$

i.e. $\Phi(x,gF)$ is a solution in the field gF. Note furthermore that, as another consequence of our approach, the invariance operator groups corresponding to a given field, and that are by definition generated by the covariant operators in the Klein-Gordon or Dirac representations that in addition leave the external field invariant, will appear now in a natural way as subgroups of the operator groups defined by (5.5) and (5.6) respectively, i.e. to ordinary representations of subgroups of $\overline{M}$.

That the Klein-Gordon and Dirac equations transform covariantly under M is not too surprising, also if one realizes that 0 and 1/2 are just the (free) spins which are in a way already of "helicity-type", but this will not necessarily be true for higher spins equations, as an arbitrary Poincaré transformation in the corresponding (free) representations will in general mix up wave components of different spin values along a given axis and belonging thus to different irreducible representations of M, this in contradiction with the covariance statement. The difficulty of the group theoretical interpretation of these higher spin equations in the presence of an external c.u. field lies in fact even deeper: as is well known, these equations are characterized by the fact that unwanted additional (free) spin components are eliminated by the so-called constraints. In the presence of a c.u. field, the very existence of the constraints becomes questionable as they have in fact lost this group theoretical meaning.

Let us, for illustration, consider a (free) spin 3/2 particle with positive mass and moving in a <u>constant uniform magnetic</u> field. Since it is a priori not possible to choose in general a reference frame where the field is along the z-axis and at the same time $p_z = 0$ and since the case (γm)(iii) (see Tables I and III) can be seen to describe tachyons states, we are in fact in case (γm) (ii), so that the (free) 3/2 - spin representation of the extended Poincaré group will split in two irreducible sub-representations of the new covariance group, characterized by the (helicity) spins $\pm$ 3/2 and $\pm$ 1/2 respectively. Now the Dirac equation with minimal coupling describes spin $\pm$ 1/2 particles and is covariant with respect to M, as we have shown in (5.10). We can thus construct, assuming a gyromagnetic factor of 1/s [26], with s the spin, an equation for the (reducible) new "particle", by means of a <u>direct sum of two Dirac equations</u> with minimal coupling, one for each irreducible component. Since the Dirac equation is covariant with respect to **M** so is trivially the new one. This splitting of a representation in two subrepresentations, as a consequence of the presence of the field, can somehow be compared to the mass zero limit in the case of a free particle: the Poincaré representations describing massive particles of spin s split in this limit in disjoint subrepresentations for massless particles of (helicity) spin s, s-1, ... (see e.g. [27] § 16). That the above constructed equation is not equivalent to the usual known equations of motion can be seen as follows: it has been shown by Velo and Zwanziger [3-4] that the 3/2 (free-) spin equations with minimal coupling in an external c.u. magnetic field (such as the Rarita-Schwinger equation for instance) are a-causal in the sense that the propagators do not vanish for space-like vectors, whereas the Dirac (as the Klein-Gordon) equation is free of a-causality and thus so is our equation, too.

The above argumentation strongly indicates that these pathological difficulties that have given rise in the last few years to an aboundant literature can possibly be reduced to a covariance problem. We think at least that it will be possible, using the explicit form of the representations we have found, to derive more precise results about these higher spins cases and this is the reason for mentioning this problem here.

Let us also remark that it follows from our results that covariant equations of motion are not necessarily analogous for various kinds of fields.However, because the Poincaré invariance of the Maxwell equations just requires such a similitude for fields related by Poincaré transformations, this is as consistent as the fact that for example a positive mass or a massless free particle do not necessarily obey analogous equations.

In the limit $F^{(o)} \longrightarrow 0$, the solutions of the equations of motion (this does not mean the equations themselves however) will of course have, for physical reasons, to vary continuously. This situation can in a way be compared to the Landau theory of second order phase transitions where the symmetry groups and the representations change discontinuously whereas the solutions remain continuous. In our case, this limiting process will of course impose conditions on the representations which have a physical meaning. It is perhaps also interesting to note in this respect that tachyons (in a c.u. magnetic field, see Table III (γm) (iii) for example) are necessarily described by finite dimensional spinors whereas in the field 0 limit all these representations will join together to give the known infinite dimensional spinors. This fact could also be used as a trick to avoid infinite spinors when one wants to introduce (virtual) tachyons states.

It is quite clear that these last remarks are not meant as a conclusion but merely as a sketch of some applications of our results and of some new open possibilities we have been lead to, by considering this problem of covariance in external e.m. fields. We have however been able to relate the Klein-Gordon and the Dirac equations, minimally coupled to the potential of an (almost) arbitrary external e.m. field, to representations of a well defined covariance group, i.e. to reintroduce and identify a relationship, well known in the free case, but which was in fact no longer necessarily present because of the introduction of the external field.

References.

[1] N. Giovannini, Chapter IV of the present treatise

[2] see for example Kursunoglu in ref. [4], p. 21

[3] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (69), 188, 2218 (69)

[4] "Troubles in the external field problem for invariant wave equations", Tracts in Math. and Nat. Sc. Vol. 4 (rev. A.S. Wightman), Gordon and Breach (1971)

[5] G.W. Mackey, Trans. Amer. Math. Soc. 85, 134 (57), Acta Math. 99, 265 (58)

[6] K.R. Parthasarathy, Comm. Math. Phys. 15, 305 (69)

[7] V. Bargmann, Ann. of Math. 59, 1 (54)

[8] R. Schrader, Fort. der Phys. 20, 701 (72)

[9] A. Guichardet, Séminaire Bourbaki 1962/63, no. 249

[10] A. Kirillov, Uspekhi Nat. Nauk. 17, 57 (62)

[11] see E.M. Stein in "High energy and elementary particles" Trieste 1965, IAEE Vienne, pp. 563-584

[12] H. Bacry, Ph. Combe and J.L. Richard, Nuovo Cim. 70A, 267 (70)

[13] N. Giovannini, Appendix 5A of the present treatise

[14] A. Artin, "Geometrical Algebra", Interscience N.Y. (1957)

[15] see for instance G.A. Pozzi, Suppl. al Nuovo Cim. 4, 37 (66)

[16] for a review, see G.W. Mackey, "Group representations", Lecture notes, Oxford University (1971)

[17] A. Janner and E. Ascher, Physica 48, 425 (70)

[18] L.C. Chen and T. Janssen, Physica 77, 290 (74)

[19] D. Simms, "Lie groups and Quantum Mechanics", Lecture notes in Math. 52, ed. Springer (1968)

[20] N.Giovannini, Proc. 3rd Int. Coll. on Group Theoretical Methods in Physics, Marseille (1974), p. 336 and Chapter I of the present treatise.

[21] E.P. Wigner, Ann. of Math. 40, 149 (39)

[22] R. Shaw and J. Lever, Comm. Math. Phys. 38, 257 (74)

[23] R. Shaw and J. Lever, Comm. Math. Phys. 38, 279 (74)

[24] E.P. Wigner, in "Group Theor. Concepts and Methods in Elementary Particle Physics", Istanbul Summer School 1962, ed. Gürsey

[25] A. Janner and T. Janssen, Physics 53, 1 (71)

[26] see e.g. F.J. Belinfante, Phys. Rev. 92, 997 (53) and P.A. Moldauer and K.M. Case, Phys. Rev. 102, 279 (56)

[27] L. Landau, E. Lifshitz, Cours de Physique Théorique, tome 4, Editions Mir, Moscou (1972).

INDUCING FROM A NORMAL NILPOTENT SUBGROUP

Introduction

The theory of induction of representations for separable locally compact topological groups, as begun in the last years of the 19th century by Frobenius [1] for the finite groups, is now quite complete, especially since the important and well known work of Wigner [2] and Mackey [3]. In the simplest and commonest case the theory provides an explicit procedure for the construction of all irreducible unitary representations of a (regular) inessential extension $G \cong N \wedge_{\phi} H$ (semidirect product) with N normal abelian, from the representations of N and of certain subgroups of the factor group H. It has also been shown by Mackey [4] that if the semidirectness or the abelian condition is left out, it becomes necessary to consider projective representations of these subgroups of H as well.

While the occuring factor systems are easy to find when N is still abelian, the problem becomes much more difficult in the general case.

In this paper we consider this problem when N is a connected nilpotent Lie group, making profit of the work of Kirillov on the structure of the irreducible unitary representations for this case [5-6]. In the first part we review briefly the theory of Kirillov on the structure of the dual $\hat{N}$ of N. For a special kind of elements of $\hat{N}$ we prove in addition some interesting and useful extra properties. We show with examples that for a large class (with respect to practical purposes) of connected nilpotent Lie groups, but not for all, all elements in $\hat{N}$ satisfy these properties. In the second part we use Mackey's theory of induction and derive then an explicit expression for the factor sets involved. For the special kind mentioned above, and if the extension is central, the result becomes especially simple.

1. The dual of N.

We first briefly review the structure of the set $\hat{N}$ of classes of irreducible unitary representations of a connected nilpotent Lie group N, as described by Kirillov [6]. For simplicity, we assume from now on that N is simply connected, the generalization giving no difficulty [5-6]. Let thus $\underline{n}$ be the Lie algebra of N, $\underline{n}'$ the dual space of $\underline{n}$ and $\mathrm{coAd}_{\underline{n}'}(N)$ the co-adjoint representation of N on $\underline{n}'$, defined for $\nu \in \underline{n}'$ by

$$(\mathrm{coAd}_{\underline{n}'}(n)\ \nu)\ (x) \overset{\mathrm{def}}{=} \nu(\mathrm{Ad}_{\underline{n}}(n)^{-1}x) \tag{1.1}$$

with $n \in N$, $x \in \underline{n}$. Let then O_ν be the orbit of an element $\nu \in \underline{n}'$, i.e. the set of all images of ν under the action (1.1) of N. Since (1.1) defines a representation of N, the set of all orbits is a partition of $\underline{n}'$. Consider then in each such orbit one arbitrary (but fixed) element ν and consider a subalgebra $\underline{\ell} \subseteq \underline{n}$ such that

$$[\underline{\ell}\ ,\ \underline{\ell}] \subseteq \mathrm{Ker}\ \nu \tag{1.2}$$

Such a subalgebra $\underline{\ell}$ is called subordinate to ν. The following map T_ν on the Lie subgroup L of N generated by $\underline{\ell}$ ($L = \exp \underline{\ell}$)

$$T_\nu(\exp x) \stackrel{\mathrm{def}}{=} \exp\{i\nu(x)\}, \quad x \in \underline{\ell} \tag{1.3}$$

is then a one dimensional irreducible unitary representation of L. This representation can then be induced to a representation $V_{\nu,\underline{\ell}}$ of N by

$$(V_{\nu,\underline{\ell}})(n)\ f(L\lambda) \stackrel{\mathrm{def}}{=} T_\nu(\lambda.n.(\lambda')^{-1})\ f(L\lambda') \tag{1.4}$$

where λ, λ' are members of an (arbitrary but fixed) set of representatives of the (right) coset decomposition of N with respect to L, with λ' fixed by the condition $\lambda.n.(\lambda')^{-1} \in L$, and $f(L\lambda)$ is a measurable and quadratic integrable function on the coset space with respect to the (quasi-) invariant (under N) measure μ and with values in the carrier space of the representation T_ν.

The following then holds

Theorem 1.1. (Kirillov)

(i) $V_{\nu,\underline{\ell}}$ is irreducible if and only if dim $\underline{\ell}$ is maximal
(ii) Any irreducible unitary representation of N can be obtained in this way, up to equivalence
(iii) $V_{\nu,\underline{\ell}} \sim V_{\nu',\underline{\ell}'}$, both irreducible, if and only if $0_\nu = 0_{\nu'}$
(iv) $\hat{N}$, the dual of N, is always of Type I

From now on, $\underline{\ell}$ will always be assumed to be a subalgebra subordinate to ν and of maximal dimension. Such a subalgebra is also called a real polarization at ν.

Let us now distinguish the following two types in $\hat{N}$:

<u>Definition:</u> The class $[\hat{n}] \in \hat{N}$ to which $V_{\nu,\underline{\ell}}$ belongs is said to be of <u>type a</u> if and only if it is possible to choose $\underline{\ell}$ ideal. Else it will be termed of <u>type b.</u>

It is easy to verify that the implied condition does not depend on the choice of a particular element on the orbit O_ν so that, together with theorem 1.1, this condition depends effectively only on the class $[\hat{n}]$ of $V_{\nu,\underline{\ell}}$.

Clearly any $\hat{N}$ contains elements of type a. We shall call <u>N to be of type a</u> if $\hat{N}$ contains only elements of type a. Let us show with examples that the class of nilpotent Lie groups which are of type a is large enough to be interesting for practical purposes, but that it does not contain every N.

<u>Examples:</u>

1) any nilpotent Lie group whose lower central series has a length less or equal to 2 is of type a. Indeed any $\underline{\ell}$ contains then the centre (else it is not maximal) and any subalgebra containing the centre is in this case ideal.

2) any nilpotent Lie group of dimension less or equal to 5 is of type a. This follows from direct checking, using the classification by Dixmier [7] of the corresponding Lie algebras. This check shows in addition that a choice may be necessary: consider for example the 5-dimensional nilpotent algebra denoted by $\mathfrak{g}_{5,3}$ in [7], with generators X_1, X_2, ... X_5 and with commutation relations

$$[X_1, X_2] = X_4$$

$$[X_1, X_4] = X_5$$

$$[X_2, X_3] = X_5$$

all other commutators being zero. Let then the linear form ν be given by $y_4 = 0 \neq y_5$, with $\nu(\Sigma\lambda_i X_i) = \Sigma\lambda_i y_i$. The subalgebra generated by X_1, X_3, X_5 is subordinate to ν (with maximal dimension) but is not an ideal whereas the subalgebra generated by X_3, X_4, X_5 satisfies both conditions.

3) any direct product of nilpotent Lie groups of type a is clearly of type a. This enlarges of course the classes of examples just mentioned.

4) not any nilpotent Lie group is of type a, contrary to what one might perhaps expect. An example of a class of type b can be given for the 10-dimensional Lie algebra of 5 x 5 matrices $\underline{\underline{A}}$, with $A_{\mu\nu} = 0$ for $\mu \geqslant \nu$. It is quite an easy and useful exercise to verify that for $\nu(\underline{\underline{A}}) \overset{\text{def}}{=} A_{15}$ there can be no subordinate $\underline{\ell}$ (of maximal dimension) which is an ideal.

We do not know however if there exists a <u>simple general</u> characterization of the class of groups of type a.

Let us mention, for nilpotent Lie groups of type a, some useful additional properties.

<u>Proposition 1.2.</u> Let N be of type a, $\nu \in O_\nu$ in $\underline{n}'$, $[\underline{\ell}, \underline{n}] \subseteq \underline{\ell}$ a corresponding ideal real polarization at ν, then

(i) $\underline{\ell}$ is his own centralizator with respect to ν, i.e. $\forall\, x \in \underline{n}$

$$\nu([\underline{\ell}\,,\ x]) \;=\; 0 \iff x \in \underline{\ell}$$

(ii) The coset space N/L is (Borel) isomorphic with the orbit of T_ν under N and is one-to-one characterized by the classes $\overline{\nu}$ of elements of O_ν which coincide with each other when restricted to $\underline{\ell}$.

Proof: (i) $\nu([\underline{\ell}, x]) = 0$ implies $\nu([\underline{\ell} + x, \underline{\ell} + x]) = 0$ and $\underline{\ell}$ ideal implies $\underline{\ell} + \mathbb{R}x$ is a subalgebra hence $\underline{\ell} + \mathbb{R}x$ is subordinate to ν. This is a contradiction to the maximality of $\underline{\ell}$ unless $x \in \underline{\ell}$. The converse is trivial.

(ii) Since $\underline{\ell}$ is ideal one may consider the following exact sequences of groups

$$1 \longrightarrow L \longrightarrow N \longrightarrow N/L \longrightarrow 1, \quad \rho, \psi \qquad (1.5)$$

with factor set $\rho \in Z^2_\psi\,(N/L, L)$ and $\psi : N/L \longrightarrow \mathrm{Aut}\, L$ defined in the natural way. One may then construct (all) irreducible representations of N by induction from the dual $\hat{L}$ of L, via the stability subgroups of N defined for a representative $\hat{\ell}$ of each class $[\hat{\ell}] \in \hat{L}$ (see section 2). For $\hat{\ell} = T_\nu$ it follows from (i) that the stabiliser is L itself (and the induced representation $V_{\nu,\underline{\ell}}$). The result follows then straightforwardly from the definition of T_ν.
Since the orbit of T_ν, the coset space N/L and a set of coset representatives are in 1-to-1 Borel correspondence with each other, we will use, or better said abuse, as is usual, the parameter λ of (1.4) to describe all these spaces.

We now turn to the general induction procedure from a normal nilpotent subgroup N.

2. General inducing from a normal nilpotent subgroup.

Mackey's theory of induced representations is a well known procedure, at least when applied to a (regular) semidirect product $G = N \wedge_\phi H$, with G separable locally compact and N abelian (ϕ denoting some given homomorphism from H to Aut(N)) [3]. Less known perhaps is the more general case where G, separable locally compact, is any (regular) extension of a group N, not necessarily abelian, by a group H, i.e. appears in the following exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1, \quad m,\phi \tag{2.1}$$

characterized by a factor set $m : H \times H \longrightarrow N$, with $m(h_1,h_2h_3)\cdot\phi(h_1)m(h_2,h_3) = m(h_1,h_2)\cdot m(h_1h_2,h_3)$, and a map $\phi : H \longrightarrow \mathrm{Aut}(N)$ satisfying

$$\phi(h_1)\ \phi(h_2) \quad = \quad \mu(m(h_1,h_2))\ \phi(h_1h_2)$$

μ being the canonical epimorphism from N to $\mathrm{In}(N)$, the group of inner automorphisms of N. The essential difference, in this more general case, is that, as shown by Mackey [4], if N is no more abelian or if the extension (2.1) does not split, no longer ordinary representations of the adequate subgroups of H have to be considered, but certain projective ones. It is the purpose of this section to calculate explicitly the factor sets which are then involved, for the case where N is a connected nilpotent Lie group. Since the dependence of m (as in (2.1)) in these factor sets is easy to find, we assume first, in order to lighten the notation, that $m = 0$.

Let us first indicate briefly, in a way which is not usual in this context but which is, in our opinion, the best adapted one, the essential steps of the explicit generalized Wigner-Mackey construction of induced representations. Let us therefore consider some arbitrary $[\hat{n}] \in \hat{N}$, the dual of N, with $\hat{n}$ a representant of this class of irreducible representations. One defines from ϕ a map $\hat{\phi}$ on H with

$$\hat{\phi}(h) \ : \ \hat{N} \longrightarrow \hat{N} \quad , \quad \hat{\phi}(h)[\hat{n}] \ \equiv \ [\hat{n}_h]$$

as follows:

$$\hat{n}_h(n) = \hat{n}(\phi(h)^{-1} n) \quad . \tag{2.2}$$

The set of all classes $\{[\hat{n}_h]\}$ generated by $\hat{\phi}$ from $[\hat{n}]$ by H is called the orbit of $[\hat{n}]$ under H and is denoted $O_{[\hat{n}]} \subseteq \hat{N}$. The action (2.2) defines directly a subgroup $H_{\hat{n}}$ of H, which we shall call the homogeneous little group, by

$$h \in H_{\hat{n}} \iff [\hat{n}_h] = [\hat{n}]$$

i.e. $h \in H_{\hat{n}}$ if and only if there exists a unitary operator $S(h) \in U(\mathcal{H}(\hat{n}))$, $\mathcal{H}(\hat{n})$ being the representation space of $\hat{n}$, such that

$$\hat{n}_h(n) = S(h)^{-1}\, \hat{n}(n)\, S(h) \quad , \quad \forall\, n \in N \tag{2.3}$$

The map $S : H_{\hat{n}} \longrightarrow U(\mathcal{H}(\hat{n}))$ is in general not a homomorphism, but, from (2.3), using that ϕ is now an homomorphism in (2.2) and that $\hat{n}$ is irreducible, it follows from Schur's Lemma that it is a projective one, satisfying thus

$$S(h_1)S(h_2) = \tau(h_1,h_2)\, S(h_1h_2) \tag{2.4}$$

h_1, $h_2 \in H_{\hat{n}}$ and $\tau(h_1,h_2)$ some factor set in U(1), the unit circle of the complex plane.

We note here that if G is not semidirect then the elements $h \in H_{\hat{n}}$ can in general no longer be identified with a subgroup of $G_{\hat{n}}$. On the other side, as we saw, ϕ is then no longer necessarily an homomorphism. The generalization of (2.4) is then easy to compute and is given by

$$S(h_1)\, S(h_2) = \tau(h_1,h_2)\, \hat{n}(m(h_1,h_2))\, S(h_1h_2) \tag{2.4'}$$

It is quite clear from the above formulas that it is in general not easy to find explicitly operators $S(h)$ satisfying (2.3) and thus the factor systems τ we shall need in the sequel. It is just the purpose of this paper to solve this problem for the more special case we are interested in.

The isotropy group $G_{\hat{n}} \subset G$ (defined as the subgroup of G leaving $[\hat{n}]$ invariant under the action $g : \hat{n}(n) \longrightarrow \hat{n}(g^{-1} n\, g)$, N being identified with its image as subgroup of G) appears as an extension (inessential if so is G) of N by $H_{\hat{n}}$, as shown in the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G_{\hat{n}} & \xrightarrow{\pi} & H_{\hat{n}} & \longrightarrow & 1, \quad 0,\phi \\ & & \| & & \downarrow{\scriptstyle(\iota)} & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi'} & H & \longrightarrow & 1, \quad 0,\phi \end{array} \tag{2.5}$$

(ι) denoting the injection monomorphism and π, respectively π', the canonical epimorphisms. We construct then in a first step a (projective) representation of $G_{\hat{n}}$ as follows. Let L be a projective representation of $H_{\hat{n}}$ with factor set ω and carrier space $\mathcal{H}(L)$ then, using (2.3) (2.4) and defining, for all $(n,h_{\hat{n}}) \in G_{\hat{n}}$,

$$(\hat{n}_S.L)(n,h_{\hat{n}}) \stackrel{def}{=} \hat{n}(n)\ S(h_{\hat{n}}) \otimes L(h_{\hat{n}}) \tag{2.6}$$

on the tensor product Hilbert space $\mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$, we get a (projective) representation of $G_{\hat{n}}$ with factor set σ, where

$$\sigma(g_1,g_2) = \tau(\pi g_1,\pi g_2) \cdot \omega(\pi g_1,\pi g_2) \tag{2.7}$$

so that, choosing $\omega = \tau^{-1}$ on $H_{\hat{n}}$, we get an ordinary representation of $G_{\hat{n}}$, with carrier space $\mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$. The last step is now the following: we decompose H in (right) cosets with respect to $H_{\hat{n}}$ with coset representatives $\{h_i \mid i \in I\}$, I some index set (Borel) isomorphic to $H/H_{\hat{n}}$, and similarly G with respect to $G_{\hat{n}}$, choosing now as coset representatives the images of the $\{h_i\}$ under a fixed section $r : H \dashrightarrow G$, the set of representatives being then given by $\{(0,h_i)\ i \in I\}$. Let now $\mu_{\hat{n}}$ be a quasi-invariant ergodic measure on $\hat{N}$, not identically zero but vanishing on all orbits outside $O_{[\hat{n}]}$. We assume, again in order to simplify the notation, that this measure is right and left invariant, the generalization being straightforward. Since we have assumed also that the action of H on $\hat{N}$ was regular (in the sense of Mackey [4]), this measure is unique (as a class) and also transitive, i.e. concentrated on the orbit [4]. Further, the coset space $G/G_{\hat{n}} \cong H/H_{\hat{n}}$ can be identified with the orbit $O_{[\hat{n}]}$ by the 1-to-1 Borel isomorphism $[\hat{n}_{h_i}] \longleftrightarrow h_i$, so that we may use, similarly as in section 1 the parametrization $\{h_i\}$ to describe both spaces. We now consider on this space a vector valued function f

$$f : h_i \dashrightarrow \mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$$

satisfying the two conditions

$$\begin{aligned} &\text{(i)} \quad (f(h_i),\varphi) \text{ is } \mu_{\hat{n}}\text{-measurable}, \ \forall\, \varphi \in \mathcal{H}(\hat{n}) \otimes \mathcal{H}(L) \\ &\text{(ii)} \quad \|f\|^2 \overset{\text{def}}{=} \int_{O_{[\hat{n}]}} \|f(h_i)\|^2 \, d\mu_{\hat{n}}(h_i) < \infty \end{aligned} \tag{2.8}$$

where scalar product and norm under the integrals are taken in $\mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$. Identifying functions equal almost everywhere, the set of functions satisfying the above conditions can be shown [3] to form a separable Hilbert space, $\mathcal{H}$, with scalar product

$$(f,g) = \int_{O_{[\hat{n}]}} (f(h_i),\ g(h_i)) \ d\mu_{\hat{n}}(h_i) \qquad .$$

The induced representation is then defined on $\mathcal{H}$ as follows: let $(n,h) \in G$, then

$$(\hat{n} \uparrow G)^L(n,h)\ f(h_i) \stackrel{\text{def}}{=} (\hat{n}_S \cdot L)((0,h_i)(n,h)(0,h_j)^{-1})\ f(h_j) \quad (2.9)$$

where h_j is the (unique) coset representative satisfying

$$h_i\ h\ h_j^{-1} \in H_{\hat{n}} \quad .$$

The following then holds, and follows from [4] for this special case:

Theorem 2.1. (Mackey) Let G, separable locally compact, be any regular (split) extension of a normal subgroup N by a group H. Consider an orbit $O_{[\hat{n}]} \subseteq \hat{N}$ and a transitive ergodic measure $\mu_{\hat{n}}$, concentrated on this orbit as described above. Then the representation (2.9) is unitary and irreducible if and only if L is unitary and irreducible. Two such representations are equivalent only if they are based on the same orbit. Moreover all irreducible unitary representations of G are obtained, up to equivalence, once and only once, when one induces once per orbit, and for each orbit one considers all projective, inequivalent, irreducible unitary ω-representations L of the corresponding $H_{\hat{n}}$, with ω satisfying the equation (2.7) where σ is taken equal to 1, and τ is determined by (2.4).

Note that the above theorem remains true when the extension is not split, with simply the condition (2.4) replaced by (2.4)'.

Up to now we have made no use of the fact that N is nilpotent and the results described above are valid for any regular (split) extension (2.1). Our purpose is now to use the results of section 1 to construct explicitly the factor sets ω.

Let thus $[\hat{n}] \in \hat{N}$, and N nilpotent. It follows from theorem 1.1 (i) that for each class $[\hat{n}]$, one can choose a representative $V_{\nu,\underline{\ell}}$ as defined in (1.4) with carrier space

$$\mathcal{H}(V) = \int_{N/L}^{\oplus} \mathcal{H}(T)\, d\mu(\lambda) \tag{2.10}$$

with T and λ as in section 1. The canonical action $\hat{\phi}$ of $h \in H$ is now given by (2.2), i.e.

$$\hat{\phi}(h)\, V_{\nu,\underline{\ell}}(n) = V_{\nu,\underline{\ell}}(\phi(h)^{-1}n) \tag{2.11}$$

defining the little group $G_{\hat{n}}$ and its factor group $H_{\hat{n}}$ as in (2.5). It follows then from (2.3), and by the definition of $H_{\hat{n}}$, that for all $h \in H_{\hat{n}}$, there exists a unitary operator S(h) such that

$$S(h)\, (\hat{\phi}(h)V_{\nu,\underline{\ell}})(n)\, S(h)^{-1} = V_{\nu,\underline{\ell}}(n)$$

and, since G is split, S(h) satisfies the condition

$$S(h_1)\, S(h_2) = \omega^{-1}(h_1,h_2)\, S(h_1h_2) \tag{2.12}$$

$\forall\, h_1,h_2 \in H_{\hat{n}}$, with the factor set $\omega^{-1}(h_1,h_2)$ we have to determine. We now construct this operator explicitly. In a first step we consider on $\mathcal{H}(V)$ an operator U(h) with

$$U(h) : \mathcal{H}(V) \longrightarrow \mathcal{H}(V')$$

where $V' \equiv V_{\nu',\underline{\ell}'}$, $\nu' = \mathrm{coAd}(h)\nu$, $\underline{\ell}' = \mathrm{Ad}(h)\underline{\ell}$. That $\underline{\ell}'$ is indeed subordinate to ν' whenever $\underline{\ell}$ is subordinate to ν follows from the fact that by (1.1) we have

$$[\underline{\ell}\,,\underline{\ell}] \subseteq \mathrm{Ker}\ \nu \Longleftrightarrow [\mathrm{Ad}(h)\underline{\ell},\ \mathrm{Ad}(h)\underline{\ell}] \subseteq \mathrm{Ker\ coAd}(h)\nu \quad .$$

Because N is normal in G we may choose now the set $\{h\lambda h^{-1}\}$, which is clearly Borel isomorphic to the set $\{\lambda\}$, to describe the coset space $N/hLh^{-1} \equiv N/L'$ on which the representation $V_{\nu',\underline{\ell}'}$ is based and we may now define the action of U(h) by

$$(U(h)f)\ (h\lambda h^{-1}) \overset{\text{def}}{=} f(\lambda) \qquad . \tag{2.13}$$

Using now the invariance of the measure μ on the coset space N/L and the definition of the scalar product in $\mathcal{H}(V)$

$$(f,g) = \int_{N/L} (f(\lambda),\ g(\lambda))\ d\mu(\lambda) \tag{2.14}$$

f, g as in (1.4), it follows from N normal in G that $\mu'(\lambda) \equiv c_h \mu(h^{-1}\lambda h)$ is invariant under N, and because this measure is unique as a class, one may choose c_h such that U(h) becomes unitary. This implies in turn that $(Uf)(h\lambda h^{-1})$ is measurable and squared integrable on N/hLh^{-1} if and only if $f(\lambda)$ is such on N/L. We now use this map to bring the representation (2.11) on a representation explicitly of the Kirillov type. We have by (1.4) and (2.13), with $g \in \mathcal{H}(V')$,

$$\begin{aligned}
&(U(h)\ V_{\nu,\underline{\ell}}(h^{-1}nh)\ U(h)^{-1}g)\ (h\lambda h^{-1}) = \\
&= (V_{\nu,\underline{\ell}}(h^{-1}nh)\ U(h)^{-1}g)\ (\lambda) = \\
&= (T_{\nu,\underline{\ell}}(l_1)\ U(h)^{-1}g)\ (\lambda_1) \qquad ,
\end{aligned} \tag{2.15}$$

with $l_1 = \lambda(h^{-1}nh)\ \lambda_1^{-1} \in L$, i.e.

$$hl_1h^{-1} = (h\lambda h^{-1})\ n(h\lambda_1 h^{-1})^{-1} \in hLh^{-1}$$

so that, using $T_{\nu,\underline{\ell}}(l_1) = T_{\nu',\underline{\ell}'}(l_1')$, with $\nu' = \text{coAd}(h)\nu$, $l_1' = hl_1h^{-1}$, $\underline{\ell}' = \text{Ad}(h)\underline{\ell}$, and the fact that T is one-dimensional, we obtain for the expression in (2.15) (that we denote by I)

$$I = (T_{\nu',\underline{\ell}'}(1_1') U(h)^{-1}g)(\lambda_1) = T_{\nu',\underline{\ell}'}(1_1')g(h\lambda_1 h^{-1})$$

$$= (V_{\nu',\underline{\ell}'}(n)\, g)(h\lambda h^{-1})$$

hence we have

$$U(h)\, V_{\nu,\underline{\ell}}(h^{-1}nh)\, U(h)^{-1} = V_{\nu',\underline{\ell}'}(n) \tag{2.16}$$

with $\nu' = \mathrm{coAd}(h)\nu$ and $\underline{\ell}' = \mathrm{Ad}(h)\underline{\ell}$.

This result is actually true for all $h \in H$, since we have made up to now no use of the isotropy condition.

Let now h be in $H_{\hat{n}}$. This implies per hypothesis, and from theorem 1.1 (iii), that ν' is in the same orbit O_ν in $\underline{n}'$ as ν (under the action of N). Thus there exists a $n \in N$, depending on h (and denoted n(h)) such that $\nu' = \mathrm{coAd}(n(h))\nu$ and $U(n(h)) = U(h)$. In other words we know there exists a n(h) such that, from (2.16)

$$V_{\nu',\underline{\ell}'}(n) = U(h)\, V_{\nu,\underline{\ell}}(n(h)^{-1}n\cdot n(h))\, U(h)^{-1}, \ \forall\, n \in N$$

and, since $V_{\nu,\underline{\ell}}$ is a representation of N

$$V_{\nu',\underline{\ell}'}(n) = U(n)\, V^{-1}_{\nu,\underline{\ell}}(n(h))\, V_{\nu,\underline{\ell}}(n)\, V_{\nu,\underline{\ell}}(n(h))\, U(h)^{-1}\ . \tag{2.17}$$

We note here that the choice of n(h) is in general not unique: one may indeed still add to it any element of the centre of the representation.

Combining now (2.17) with (2.16), and with the definition (2.3) of S(h) we have found however an explicit expression for the latter:

$$S(h) = V_{\nu,\underline{\ell}}(n(h)) \tag{2.18}$$

which is unitary.

If G is not semidirect the same intertwining operator may still be used, corresponding then to the action of the element $(0,h)$ of $G_{\hat{n}}$, in the extension notation. Since H can then no longer be identified with a subgroup of G, the action of H on $\underline{n}'$ is no longer a representation: we have then

$$\mathrm{coAd}(0,h_1)\ \mathrm{coAd}(0,h_2) \;=\; \mathrm{coAd}((m(h_1,h_2),\ 1))\ \mathrm{coAd}(0,h_1h_2) \quad (2.19)$$

corresponding to the analogous change in (2.4)'. Taking these modifications into account we may now formulate a first general result for our factor systems:

Proposition 2.2. Let G, separable locally compact, be any regular extension of a (connected) nilpotent Lie group N by a group H. Then given $[\hat{n}] \in \hat{N}$, $V_{\nu,\underline{\ell}} \in [\hat{n}]$, the intertwining operator $S(h) \equiv S((0,h))$ of (2.3) and the factor system ω of (2.7) on $H_{\hat{n}}$ needed for the Wigner-Mackey generalized induction procedure are respectively given by

$$S((0,h)) \;=\; V_{\nu,\underline{\ell}}(n(h))$$

with $n(h) \in N$ such that $\mathrm{coAd}(n(h))\nu = \mathrm{coAd}(h)\nu$, and

$$\begin{aligned}\omega(h_1,h_2)\ \mathbb{1} \;&=\; \left[S((0,h_1)).S((0,h_2)).S((0,h_1)(0,h_2))^{-1}\right]^{-1} \\ &=\; V_{\nu,\underline{\ell}}(m(h_1,h_2).V_{\nu,\underline{\ell}}(n(h_1h_2))V^{-1}_{\nu,\underline{\ell}}(n(h_1)n(h_2))\end{aligned} \quad (2.20)$$

This factor set can thus now be computed straightforwardly. However, when the representation class of $V_{\nu,\underline{\ell}}$ is of type a with $\underline{\ell}$ ideal, and for central extensions (definition given below), the result becomes rather simpler.

Indeed we note that if $\underline{\ell}$ is ideal the choice of $n(h)$ is unique up to the subgroup of N leaving the orbit of $T_{\nu,\underline{\ell}}$ point per point invariant, i.e. by Proposition 1.2, up to an element of L. In other words for a given $h \in H_{\hat{n}}$ the set of n(h) satisfying (2.17) corresponds uniquely to an element of the factor group N/L.

Let us therefore consider again the exact sequence of groups (1.5):

$$1 \longrightarrow L \longrightarrow N \longrightarrow N/L \longrightarrow 1, \quad \rho,\psi$$

and suppose first again that G is semidirect. We have then (identifying for simplicity L with its image in N under (ι), the canonical injection, and the elements of N/L with their image under a fixed section $s : N/L \dashrightarrow N$):

$$\begin{aligned} S(h_1)\, S(h_2)\, S(h_1h_2)^{-1} &= V_{\nu,\underline{\ell}}(n(h_1))\, V_{\nu,\underline{\ell}}(n(h_2))\, V^{-1}_{\nu,\underline{\ell}}(n(h_1h_2)) \\ &= V_{\nu,\underline{\ell}}(\rho(n(h_1),n(h_2))) \end{aligned} \quad .$$

Hence, with (2.20) and the definition of $V_{\nu,\underline{\ell}}$

$$\omega(h_1,h_2) = T^{-1}_{\nu,\underline{\ell}}(\rho(n(h_1),n(h_2))) \quad .$$

For example if N is abelian, L = N, so that the extension (1.5) is trivially split. Thus ρ and hence ω are indeed trivial. We can now easily generalize the above result for the case where the extension (2.1) is not inessential but central, i.e. characterized by a factor set m in the centre of N. Indeed $V_{\nu,\underline{\ell}}(m(h_1,h_2))$ in (2.20) is then necessarily a phase, this corresponds also to the fact that m in the centre of N implies $m \subseteq L$, for any orbit. On the other side, $m \subseteq L$ implies also that $n(h_1) \cdot n(h_2)$ is in the same coset of N with respect to L as $n(h_1h_2)$, $\forall\, h_1,h_2 \in H_{\hat{n}}$. Hence we have the following

<u>Proposition 2.3.</u> Let G, separable locally compact, be a central (regular) extension of a (connected) nilpotent Lie group N of type a

by a group H. Then given $[\hat{n}] \in \hat{N}$, $V_{\nu,\underline{\ell}} \in [\hat{n}]$ (with $\underline{\ell}$ ideal) the factor system ω described above is given, up to equivalence, by

$$\omega(h_1,h_2) = T_{\nu,\underline{\ell}}(m(h_1,h_2))\, T^{-1}_{\nu,\underline{\ell}}(\rho(n(h_1),n(h_2))) \quad . \qquad (2.21)$$

Again, for N abelian this gives the known result back, ρ being trivial.

The Proposition 2.3 clearly remains true if N is not of type a and/or the extension is non central, but only for the representations $\hat{n}$ of N which satisfy the two conditions

(i) $[\hat{n}]$ is of type a

(ii) $m(H_{\hat{n}},H_{\hat{n}}) \subseteq L$.

This problem has been developed for the purpose of a specific physical situation [8]. We refer to this paper for a practical application of these results. Let us just give here a short example.

Example. Let $G = N \wedge_{\phi} H$ (semidirect) where N is the 9-dimensional nilpotent Lie group with infinitesimal generators P_μ, X_μ and I (μ = 0,1, 2,3) satisfying the following commutation relations

$$[P_\mu, X_\mu] = g_{\mu\nu} . I \qquad\qquad \mu,\nu = 0,1,2,3$$

where $g_{\mu\nu}$ is some invertible "metric tensor". All other commutators vanish. For example one may have $H = SL(2,\mathbb{C})$ with $g_{\mu\nu}$ the Minkovski metric and $\phi(SL(2C))$ the usual action on momentum and position operators.

Since the lower central series has length 2, N is of type a; let now $\nu \in \underline{n}'$. These are obviously two cases:

(i) $I \in \operatorname{Ker} \nu$. Then $\underline{\ell} = \underline{n}$ and ρ is trivial hence ω is trivial, too (by (2.21), m being zero).

(ii) $I \notin \operatorname{Ker} \nu$. The orbits O_ν are then 8-dimensional and characterized by the value of ν on I. The maximal dimension of $\underline{\ell}$ is thus 5, since [6], if $[\ell, \ell] \subseteq \operatorname{Ker} \nu$

$$\operatorname{Max} (\dim \underline{\ell}) = \dim \underline{n} - \tfrac{1}{2} \dim O_\nu$$

$\underline{\ell}$ can obviously always be chosen as the (abelian) subalgebra generated by I, P_0, P_1, P_2 and P_3, which is an ideal. The extension (1.5) is then split hence ρ and thus ω are trivial.

This nilpotent Lie group behaves thus in the induction procedure of a semidirect product always like an abelian one [1)], i.e. the procedure reduces to the usual case.

---

1) For the special case of $H = SL(2,\mathbb{C})$ mentioned above this result has already been obtained by Angelopoulos [9], in a paper considering the same problem, but from an opposite point of view: this author explored namely conditions that could be imposed on H (in the semi-direct product case) so to keep ω inessential and proved that for H semisimple, ω was not in general inessential but that it always could be chosen real (hence equal to $\pm$ 1).

References

[1] G. Frobenius, Sitz. Preus. Akad. Wiss. 1898, p.501

[2] E.P. Wigner, Ann. of Math. 40 (1939), 149

[3] for a review of the theory see for example G.W. Mackey, Lecture notes, U.P. Oxford, 1971

[4] G.W. Mackey, Acta Math. 99 (1958), 265

[5] A. Guichardet, Séminaire Bourbaki, 1962/63, no. 249

[6] A. Kirillov, Uspekhi Mat. Nauk, 17 (1962), 57

[7] J. Dixmier, Can. J. Math. 10 (1958), 321

[8] N. Giovannini, Chapter V of the present treatise

[9] E. Angelopoulos, Ann. Inst. H. Poincaré, 18 (1973), 39

SUMMARY

The present study contains a general group theoretical analysis of the problem of a charged massive particle moving in an (arbitrary) classical external e.m. field. This analysis is essentially based on the space-time symmetry properties of e.m. fields and of e.m. field equations, and on the fact that the considered equations of motion depend on the field only through a potential.

A space-time transformation that leaves an e.m. potential invariant is clearly also a symmetry for the corresponding field but the converse is, in general, not true. It is, however, always possible to combine the space-time symmetries of the field with (compensating) gauge transformations in such a way that the operators induced by the combined transformations commute with the operators of the corresponding equations of motion, and to construct in this way invariance operator groups for the equations of motion.

<u>In the first part</u> of the present work we derive explicitly these invariance operator groups for (almost) arbitrary external fields, using general properties of those space-time symmetry groups that can occur for e.m. fields and a fixed choice of gauge for the corresponding potentials. Other choices of gauge give rise to isomorphic groups. In this way the explicit structure of these invariance operator groups can be analysed and discussed in detail.

This program is pursued in <u>Chapter I</u> for the relativistic case (Klein-Gordon and Dirac equations), and in <u>Chapter II</u> for the Galilean case and for the case where only Euclidian transformations with time-translations and time-reversals are allowed (Schrödinger equation). The validity of Galilean symmetry transformations for e.m. fields (in an exact or in an approximate sense) is also discussed. As a further result, we show also that there exists quite a large class of

fields that give rise to invariance operator groups that are not of Type I. Such groups are not only mathematically very pathological and hard to handle, they are also physically quite difficult to interpret and give rise to important problems.

In this first part we also do not (as is often the case in such problems) restrict ourselves only to continuous or only to discrete symmetries for the fields concerned. We show, on the contrary, that the interrelations between these two kinds of symmetries may be important and lead to non-trivial consequences.

In Chapter III we exhibit, using the preceding results (but for the Type I cases only) the relations between these invariance operator groups and the solutions of the corresponding equations of motion, and we indicate a straightforward method for computing the irreducible representations of these groups, especially in the non-trivial case where the compensating gauge transformations generate a factor system inequivalent to zero. In particular we show how the well known Bloch ansatz can then be generalized to the case of an arbitrary periodic field in space and time that includes a non zero constant uniform component.

In the second part we analyse the concept of covariance in the presence of an external e.m. field and its relationship to the equations of motion themselves. Relativistic equations of motion for a free particle are covariant under the Poincaré group and are uniquely related to representations of this group (defining at the same time the group theoretical notion of elementary particles). This relationship, however, does no longer necessarily hold in the presence of an external field. On the other hand, the invariance operator groups discussed previously are not, in general, subgroups of the space-time covariance operator groups that characterize the corresponding equations of motion, because of the presence of the gauge transformations and of possibly non-trivial factor systems. Invariance (that relates identical physical system) is, however, necessarily a particular case of covariance (that relates equivalent physical systems): just as there is an invariance transformation for any space-time symmetry of the field, there ought to be a covariance transformation for any space-time symmetry of the field equations.

In Chapter IV such a general covariance group is derived explicitly. This group is obtained independently of any specific equation of motion, essentially on the basis of the Poincaré invariance of the Maxwell equations. In Chapter V we analyse the PUAIR (projective unitary/antiunitary irreducible representations) of this group and we connect them, in a way analogous to the free case, to a group theoretical concept of elementary particles in interaction with an external e.m. field and to the concept of covariant equations of motion. It follows from this analysis that important physical quantities like the spin may then be given a natural group theoretical meaning that may differ from the (usual) one of the free case. A further result is that the Klein-Gordon and Dirac equations with minimal coupling are covariant under this new group, i.e. characterize representations of this group, but that this does not need to be the case for the usual higher spin equations with minimal coupling. It is also shown how this fact leads to a possible explanation of the a-causality troubles characterising these equations. As another consequence of this approach, the invariance operator groups investigated in the first part of the present study now appear in a very natural way as subgroups of the covariance operator groups corresponding to the equation of motion considered - i.e. as ordinary (and no longer projective) representations of subgroups of these covariance operator groups.

SAMENVATTING

Deze studie bevat een algemene groepentheoretische analyse van het probleem van een geladen massief deeltje dat beweegt in een (willekeurig) klassiek uitwending e.m. veld. Deze analyse is hoofdzakelijk gebaseerd op de ruimte-tijd symmetrie-eigenschappen van e.m. velden en van e.m. veld vergelijkingen, en op het feit dat de beschouwde bewegingsvergelijkingen alleen via een potentiaal van het veld afhangen.

Een ruimte-tijd transformatie die een e.m. potentiaal invariant laat is duidelijk ook een symmetrie voor het corresponderende veld maar het omgekeerde is, in het algemeen, niet waar. Het is echter altijd mogelijk de ruimte-tijd symmetrieën van het veld te combineren met (compenserende) ijktransformaties, zodanig dat de door de gecombineerde transformaties geïnduceerde operatoren commuteren met de operatoren van de corresponderende bewegingsvergelijking, en op die manier invariantie-operatorengroepen te construeren voor die bewegingsvergelijkingen.

In het eerste gedeelte van deze dissertatie, leiden we op een expliciete manier deze invariantie groepen af voor (vrijwel) willekeurige uitwendige velden, gebruikmakend van algemene eigenschappen van de ruimte-tijd symmetriegroepen die voor e.m.velden kunnen voorkomen en van een vaste keuze van ijking voor de corresponderende potentialen. Andere keuzen van ijking geven aanleiding to isomorfe groepen. Op die manier kan de expliciete struktuur van deze invariantie-operatorengroepen in detail worden geanalyseerd en bediscussieerd.

Dit program wordt in Hoofdstuk I uitgevoerd voor het relativistische geval (Klein-Gordon en Dirac vergelijkingen), en in Hoofdstuk II voor het Galilei geval en voor het geval waar alleen Euclidische transformaties, tijd-translaties en tijdomkeer zijn toegestaan (Schrödinger vergelijking). De geldigheid van Galilei transformaties voor e.m. velden (in een exacte of in een benaderde zin) wordt ook besproken. Als een verdere resultaat, laten we ook zien dat er een vrij brede klasse van velden bestaat die aanleiding geven tot invariantie-operatorengroepen die niet van Type I zijn. Zulke groepen zijn niet alleen mathematisch zeer pathologisch en moeilijk te behandelen, ze zijn ook fysisch vrij moeilijk te interpreteren en geven aanleiding tot belangrijke problemen.

In dit eerste gedeelte beperken wij ons verder niet tot alleen continue of diskrete symmetrieën voor de velden (zoals vaak gedaan wordt in zulke problemen). In tegendeel, we laten ook met voorbeelden zien, dat de interrelatie tussen deze twee soorten van symmetrieën belangrijk kan zijn en consequenties kan hebben die niet triviaal zijn.

In Hoofdstuk III tonen wij aan, gebruik makend van de voorafgaande resultaten (maar alléén voor de Type I gevallen), de samenhang tussen deze invariantie-operatorengroepen en de oplossingen van de corresponderende bewegingsvergelijkingen en we geven een eenvoudige direkte methode aan voor de berekening van de irreducibele representaties van deze groepen, in het bijzonder in het geval waar de compenserende ijktransformaties een factor systeem opleveren dat niet equivalent met nul is. In het bijzonder laten we dan zien hoe de welbekende Bloch ansatz gegeneraliseerd kan worden voor het geval van een willekeurig, in ruimte en tijd periodiek veld dat ook een niet-verdwijnende konstante uniforme komponente heeft.

In het tweede gedeelte analyseren wij dan het concept van covariantie in de aanwezigheid van een uitwendig e.m. veld en diens samenhang met de bewegingsvergelijkingen zelf. Relativistische bewegingsvergelijkingen voor vrije deeltjes zijn covariant onder de Poincaré groep en zijn op een unieke manier verbonden aan representaties van deze groep (tegelijkertijd groepertheoretisch het begrip van elementair deeltje definiërend). Deze betrekking bestaat echter niet noodzakelijk meer in de aanwezigheid van een uitwendig veld. Aan de andere kant zijn de eerder besproken invariantie-operatorengroepen in het algemeen geen ondergroepen van de ruimte-tijd covariantie-operatorengroepen die de corresponderende bewegingsvergelijkingen karakteriseren, dit als gevolg van de aanwezigheid van ijktransformaties en van mogelijkerwijze niet-triviale factorsystemen. Invariantie (die identieke fysische systemen verbindt) is echter noodzakelijkerwijze een speciaal geval van covariantie (die equivalente fysische systemen verbindt): zoals er een invariantie-transformatie is voor elke ruimte-tijd symmetrie van het veld, moet er een covariantie-transformatie bestaan voor elke ruimte-tijd symmetrie van de veld-vergelijkingen.

In Hoofdstuk IV wordt zo'n algemene covariantie groep expliciet afgeleid. Deze groep wordt onafhankelijk van enige specifieke bewegingsvergelijking verkregen, hoofdzakelijk op basis van de Poincaré invariantie van de Maxwell vergelijkingen. In Hoofdstuk V analyseren wij dan de PUAIR (projectieve unitaire/antiunitaire irreducibele representaties) van deze groep en wij verbinden ze, op een analoge manier als in het vrije geval, met het groepentheoretische concept van elementair deeltje in interactie met een uitwendig e.m. veld, en met het concept van covariante bewegingsvergelijkingen. Uit deze analyse volgt dat belangrijke fysische begrippen, zoals de spin, een natuurlijke groepentheoretische betekenis krijgen die van de gebruikelijke (vrij geval) kan afwijken. Verder laten we zien dat de Klein-Gordon en de Dirac vergelijkingen met minimale koppeling covariant zijn onder deze nieuwe groep, i.e. zij karakteriseren representaties van deze groep; dat hoeft echter niet het geval te zijn voor de gebruikelijke vergelijkingen bij hogere spin met minimale koppeling. We laten ook zien hoe dit feit kan worden verbonden met een mogelijke verklaring van de moeilijkheden van a-causaliteit die deze vergelijkingen karakteriseren. Een andere konsekwentie van deze analyse is dat de in het eerste gedeelte van dit proefschrift geanalyseerde invariantiegroepen nu op een natuurlijke wijze verschijnen als ondergroepen van de nieuwe covariantie-operatorengroepen die corresponderen met de betreffende bewegingsvergelijking -i.e. als gewone (en niet langer projectieve) representaties van ondergroepen van deze covariantie operatorengroepen.

Curriculum vitae

De auteur van dit proefschrift werd op 27 november 1947 te Lausanne (Zwitserland) geboren. Na de lagere school bezocht hij de College Saint-Michel te Fribourg waar hij in 1966 het eindexamen behaalde. Hij begon zijn wis- en natuurkunde studie aan de Eidgenössische Technische Hochschule te Zürich en studeerde daar in mei 1971 af, met als hoofdvak theoretische fysica. Na 1 jaar als fellowship, is hij sinds september 1972 werkzaam als wetenschappelijk medewerker aan het Instituut voor Theoretische Natuurkunde (vaste stof) aan de Katholieke Universiteit te Nijmegen.

STELLINGEN

1. De factorsystemen, die nodig zijn voor het door inductie bepalen van de irreducibele unitaire representaties van een groep G die een centrale uitbreiding is van een groep N met een groep H, kunnen geschreven worden als producten van twee onafhankelijke factorsystemen, op zo'n manier dat één triviaal is als N abels is, en het andere als de uitbreiding inessentieel is. Een uitbreiding wordt daarbij centraal genoemd als het factorsysteem $m: H \times H \rightarrow N$ alléén waarden in het centrum van N heeft

   G. W. Mackey, Acta Math. *99* (1958), 265

2. Zij G een gesloten ondergroep van de Poincarégroep P, K zijn puntgroep, G_0 zijn componente van de eenheid. Als K eindig vele samenhangende componenten heeft en als het translatiegedeelte van G_0 samenhangend is, dan is G isomorf met een semidirect product van G_0 met een discrete ondergroep van P.

3. Het algemene standpunt van Mackey, dat de verandering in tijd van een fysisch systeem kan worden beschreven met een semigroep U_t met één parameter, eist wel degelijk „significant physical prerequesites"

   G. W. Mackey, The Mathematical Foundations of Quantum Mechanics, eds Benjamin, 1963.

4. De uitdrukking „strictly ergodic measure" is misleidend, zoals ongelukkigerwijze te veel nomenclatuur in de fysica, aangezien het juist het tegendeel betekent van wat het suggereert.

5. In tegenstelling tot de bewering van sommige auteurs, bestaan er nilpotente Lie groepen die niet van Type I zijn, zoals de groep van tripels $\{(m,n,\alpha), m,n \in Z, \alpha \in R\}$ met product $(m,n,\alpha)(m',n',\alpha') = (m+m', n+n', \alpha + \alpha' + mn')$.
   Deze groep kan worden geidentificeerd met een ondergroep van de Galileigroep in twee dimensies.

   Zie bijv. A. J. Coleman in „Group Theory and Its Applications" vol. 1, ed. by E. M. Loebl, Academic Press, New York and London, 1968, p. 57

6. Het is verbazingwekkend dat begrippen zoals presymmetry (of het equivalente (!) post symmetry) in de fysica kunnen worden ingevoerd zonder definitie.

   H. Eckstein, Phys. Rev. *153* (1967), 1397
   J. M. Lévy-Leblond, Ann. Phys. *57* (1970), 481.

7. Voor richtingen in een symmetrievlak kan de Greense functie van een oneindig cubisch elastisch medium, en al zijn afgeleiden, analytisch worden uitgedrukt.

N. Giovannini and J. Muggli, J Phys. C. *5* (1972), 374.

8. Door een storingsrekening toe te passen op de massa kunnen de begrippen van Brillouinzones en Fermioppervlakken voor zuiver periodieke velden in ruimte en tijd worden gegeneraliseerd. In het geval van het veld van een monochromatische vlakke golf, zijn de (1-dimensionale) grenzen van deze gegeneraliseerde Brillouinzones allemaal ontaard en liggen ze op de lichtkegel langs de k-vector van de vlakke golf. De exacte oplossingen van het probleem divergeren op deze grenzen.

D. M. Volkow, Z. Phys. *94* (1935), 250
A. Janner and T. Janssen, Physica *60* (1972), 292.

9 Uit de afwezigheid van enige Poincarétransformatie die, zelfs maar in een benaderde zin, de twee situaties op elkaar afbeeldt die leiden tot de z.g. tweelingparadox (in één situatie gaat één van beiden op reis, in de andere vertrekken ze allebei en in tegengestelde richtingen), volgt dat die twee situaties niet relativistisch equivalent zijn, zelfs niet in een benaderde zin, en dat de paradox kan worden begrepen door simpele symmetriebeschouwingen.

10. Het zou onjuist zijn aan te nemen dat een wiskundig object, zoals de quantummechanica, zijn eigen interpretatie zou kunnen bevatten. Men moet dus zorgvuldig onderscheid maken tussen de betekenis van realiteit als verbonden aan het object of als verbonden aan zijn interpretatie. De titel „Are Quanta Real", bijvoorbeeld, voldoet niet aan deze eis

J M. Jauch, „Are Quanta Real. A Galilean Dialogue",
Bloomington Indiana U.P., 1973.

11. Het beleid van de overheden dat een concentratie van het geld bevordert voor die takken van wetenschap die bepaalde, direct nuttige problemen beschouwen (zoals energie, kanker enz.) berust op een slechte kennis van de geschiedenis van de wetenschappen en van de wetenschappelijke ontdekkingen.

12. Een democratisch systeem, dat gebaseerd is op de mening van de meerderheid, kan pas functioneren als ook rekening wordt gehouden met de mening van de minderheden.

Nijmegen, 7 october 1976 N. Giovannini